Solvable groups with a given solvable length, and minimal composition length

S. P. GLASBY

Abstract. Let $c_{\mathbf{S}}(d)$ denote the minimal composition length of all finite solvable groups with solvable (or derived) length d. We prove that:

d	0	1	2	3	4	5	6	7	8
$c_{\mathbf{S}}(d)$	0	1	2	4	5	7	8	13	15

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1. Introduction

Let $c_{\mathbf{N}}(d)$ (resp. $c_{\mathbf{S}}(d)$) denote the minimal composition length of a finite nilpotent group (resp. solvable group) with solvable length d. Burnside [1] knew that $c_{\mathbf{N}}(0) = 0$, $c_{\mathbf{N}}(1) = 1$, $c_{\mathbf{N}}(2) = 3$, and $c_{\mathbf{N}}(3) = 6$. It is shown in [3] and [4] that $c_{\mathbf{N}}(4) = 13$. Exact values of $c_{\mathbf{N}}(d)$ for $d \geqslant 5$ are unknown. P. Hall showed that $2^{d-1} + d - 1 \leqslant c_{\mathbf{N}}(d) \leqslant 2^d - 1$, see [10, 9]. For $d \geqslant 4$, Evans-Riley et al. [4] improved the upper bound to $2^d - 2$, and the author (unpublished notes, 1993) improved the lower bound to $2^{d-1} + d + 1$. Mann [12] and Schneider [15] further improved the lower bound to $2^{d-1}+2d-4$ and $2^{d-1}+3d-10$ respectively. Upper bounds are proved by producing specific examples. Constructing groups of order p^n and solvable length $\lfloor \log_2 n \rfloor + 1$ appears difficult, and doing so for minimal n requires prescience. Such constructions commonly do not work for all primes.

Let $C_{\mathbf{N}}(d)$ (resp. $C_{\mathbf{S}}(d)$) denote the set of all isomorphism classes of finite nilpotent groups (resp. solvable groups) having solvable length d, and minimal composition length. We shall blur the distinction between a group G, and the isomorphism class $[G]$ that it represents. Accordingly, we write $G \in C_{\mathbf{N}}(d)$ (resp. $G \in C_{\mathbf{S}}(d)$) as an abbreviation for the phrase "G is a nilpotent group (resp. solvable group) with solvable length d, and minimal composition length." For $G \in C_{\mathbf{N}}(d)$ or $C_{\mathbf{S}}(d)$, $G^{(d-1)}$ is the unique minimal normal subgroup of G (Lemma 1(a)). [Recall that the derived series for G is defined recursively by $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geqslant 0$, and the *solvable* or *derived length* of

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a solvable group G is the minimal value of d such that $G^{(d)} = 1$.] A major difficulty in studying groups $G \in \mathrm{C}_{\mathbf{N}}(d)$ is that if $d > 1$, then $G/G^{(d-1)}$ never lies in $G \in \mathrm{C}_{\mathbf{N}}(d-1)$. The reason that we have made so much progress in the solvable case is that if $G \in \mathrm{C}_{\mathbf{S}}(d)$, then $G/G^{(k)}$ is commonly an element of $\mathrm{C}_{\mathbf{S}}(k)$ for large k less than d.

There is an analogy between "minimal composition length" groups and p-groups of maximal class. The latter may be viewed as having a given nilpotency class c, and minimal composition length. This class of groups is amenable to inductive study as if G has maximal class c, then $G/\gamma_c(G)$ has maximal class $c-1$. Maximal class groups have been well studied, see for example [10], III §14, and [18].

We abbreviate the composition length of a solvable group G by $c(G)$, and its solvable (or derived) length by $d(G)$. If $|G| = p_1^{k_1} \cdots p_s^{k_s}$, where the p_i are distinct primes, then $c(G) = k_1 + \cdots + k_s$. It is clear that

$$\mathrm{c}_{\mathbf{S}}(d) + 1 \leqslant \mathrm{c}_{\mathbf{S}}(d+1) \leqslant 2\mathrm{c}_{\mathbf{S}}(d) + 1 \qquad (d \geqslant 0),$$

where the upper bound is obtained by considering the wreath product $G \operatorname{wr} C_2$ where $G \in \mathrm{C}_{\mathbf{S}}(d)$. The above inequalities imply that $d \leqslant \mathrm{c}_{\mathbf{S}}(d) \leqslant 2^d - 1$. We show in the next paragraph that $\mathrm{c}_{\mathbf{S}}(d)$ grows exponentially, and is considerably less than $\mathrm{c}_{\mathbf{N}}(d)$ for large d. For example, $17 \leqslant \mathrm{c}_{\mathbf{S}}(10) \leqslant 24$ and $532 \leqslant \mathrm{c}_{\mathbf{N}}(10) \leqslant 1022$. [M.F. Newman (pers. comm.) can show that $\mathrm{c}_{\mathbf{N}}(10) \leqslant 832$, and the author can show $20 \leqslant \mathrm{c}_{\mathbf{S}}(10)$.]

If G is the r-fold permutational wreath product $H \operatorname{wr} \cdots \operatorname{wr} H$ where $H = S_4$, then $|G| = |H|^{1+4+\cdots+4^{r-1}}$. Therefore

$$c(G) = c(H)(4^r - 1)/3 < (4/3) \cdot 4^r, \qquad \text{and} \qquad d(G) = 3r.$$

This proves that $\mathrm{c}_{\mathbf{S}}(d) < (4/3) \cdot 4^{d/3}$ when d is a multiple of 3. Since $9^{1/5} < 4^{1/3}$, a sharper bound is obtained by taking H to be the primitive subgroup $\mathrm{GL}_2(3) \ltimes C_3^2$ of S_9. Then $d(H) = 5$ and $c(H) = 7$, so $c(G) = 7(9^r - 1)/8 < (7/8) \cdot 9^r$. Thus $\mathrm{c}_{\mathbf{S}}(d) < (7/8) \cdot 9^{d/5}$ when d is a multiple of 5. Lower bounds for $\mathrm{c}_{\mathbf{S}}(d)$ require more work. It is shown in Theorem 8 of [5] that a solvable group G with $d(G) = d$ and $c(G) = n$ satisfies

$$d \leqslant \alpha \log_2 n + 9 \qquad \text{where} \qquad \alpha = 5 \log_9 2 + 1.$$

The smallest value of n satisfying the above inequality is $\mathrm{c}_{\mathbf{S}}(d)$, and so $2^{(d-9)/\alpha} \leqslant \mathrm{c}_{\mathbf{S}}(d)$. Since $0.088 < 2^{-9/\alpha}$, $1.3 < 2^{1/\alpha}$ and $9^{1/5} < 1.56$, we see that

$$(0.088)(1.3)^d < \mathrm{c}_{\mathbf{S}}(d) < (7/8)(1.56)^d \qquad (d > 0)$$

where the upper bound holds when d is a multiple of 5.

In our proof that $\mathrm{c}_{\mathbf{S}}(8) \geqslant 15$, for example, we learn enough about the structure of putative groups with $d(G) = 8$ and $c(G) = 15$ in order to construct them. Indeed, with more attention to detail we could determine a complete and irredundant list of isomorphism classes in $\mathrm{C}_{\mathbf{S}}(d)$ for $d \leqslant 8$. This requires great care as it is all to easy to omit an isomorphism class, or to list the same class twice. In this paper we fall short of this aim, however, the isomorphism problem is solved for $d \leqslant 6$ in the preprint [6].

The groups we list in $\mathrm{C}_{\mathbf{S}}(d)$, $d \leqslant 8$, have the property that their lattice of normal subgroups is a chain. The class of such groups, which we call *normally uniserial*, is closed under quotients and hence suited to inductive study. Moreover, if $M > N$ are normal subgroups of a normally uniserial group, then M/N is a *characteristically uniserial* group, i.e. its lattice of characteristic subgroups is a chain. Clearly, simple groups are normally (and hence characteristically) uniserial. In [5] the author constructs a remarkable group $G = \mathrm{GL}_2(3) \ltimes 3^{2+1} \ltimes 2^{6+1} \ltimes 3^{8+1}$ of order $2^{11}3^{13}$ with solvable length 10. (A more systematic construction of G is given in [7] where it is shown to be the derived 10 quotient of an infinite pro-$\{2,3\}$ group.) G is normally uniserial. I was surprised to learn that G is a maximal subgroup of the sporadic simple group Fi_{23} of order $2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, see [2], p. 177. Indeed, G has the property that $G/G^{(d)} \in \mathrm{C}_{\mathbf{S}}(d)$ for $d = 0, 1, 2, 3, 4, 5, 6, 8$, and very likely also for $d = 10$. For the purposes of this paper it is useful to understand the group $G/G^{(8)} = \mathrm{GL}_2(3) \ltimes 3^{2+1} \ltimes 2^{6+1}$ which is described in [13, 7]. Certain groups in $\mathrm{C}_{\mathbf{S}}(d)$, $d \leqslant 6$, have finite presentations with deficiency zero, see [8] for details.

2. The case $d \leqslant 6$

In this section we determine the solvable groups in $G \in \mathrm{C}_{\mathbf{S}}(d)$ for $d \leqslant 6$. That is, we determine solvable groups with a given solvable length $d \leqslant 6$, and minimal composition length subject to this constraint. We shall determine sufficient structure of these groups in order to compute additional values of $\mathrm{c}_{\mathbf{S}}(d)$. We stop short of classifying the groups up to isomorphism. The determination of $G \in \mathrm{C}_{\mathbf{S}}(d)$ for $d \leqslant 6$ is influenced by the elementary fact that a metacyclic group is never the derived subgroup of a group. This fact dates back to [20], Satz 9, p. 138.

Lemma 1. (a) *If $G \in \mathrm{C}_{\mathbf{S}}(d)$, then $G^{(d-1)}$ is the unique minimal normal subgroup of G.*

(b) *Let G be a solvable group with a unique minimal normal subgroup. Let $P = \mathrm{O}_p(G)$ be nontrivial, and suppose that $|P/\Phi(P)|$*

equals p^r. *Then* $\mathrm{O}_{p'}(G) = 1$, *and* G/P *is isomorphic to a completely reducible subgroup of* $\mathrm{GL}_r(p) \cong \mathrm{Aut}(P/\Phi(P))$.

(c) *If* $2 \leqslant i < d(G)$, *then* $G^{(i-1)}/G^{(i)}$ *and* $G^{(i)}/G^{(i+1)}$ *are not both cyclic. In particular,* $c(G^{(i)}/G^{(i+2)}) \geqslant 3$ *for* $1 \leqslant i < d(G) - 1$.

(d) *Let* $1 \leqslant i < d(G)$ *and let* $G^{(i-1)}/G^{(i)}$ *be cyclic, and the unique minimal normal subgroup of* $G/G^{(i)}$. *Then* $G/G^{(i)}$ *acts faithfully as a group of automorphisms of* $G^{(i)}/G^{(i+1)}$, *and* $G/G^{(i+1)}$ *is a split extension of* $G^{(i)}/G^{(i+1)}$ *by* $G/G^{(i)}$. *Moreover,* $G^{(i-1)}/G^{(i)}$ *has order coprime to* $|G^{(i)}/G^{(i+1)}|$ *and acts fixed-point-freely.*

(e) *Suppose that* $i \geqslant 2$, $c(G^{(i-1)}/G^{(i)}) = 2$ *and* $c(G^{(i)}/G^{(i+1)}) = 1$. *Then* $G^{(i-1)}/G^{(i+1)}$ *is an extraspecial group of order* p^3.

Proof. (a) Let N be a nontrivial normal subgroup of G. If $G^{(d-1)} \not\leqslant N$, then G/N has solvable length d, and smaller composition length. Since $G \in \mathrm{C}_{\mathbf{S}}(d)$, this is impossible. Thus $G^{(d-1)} \leqslant N$, as desired.

(b) The order of the unique minimal normal subgroup is a power of some prime, say p, and $\mathrm{O}_{p'}(G) = 1$. By a result of Hall and Higman [10], VI§6.5, $C_{G/\Phi(P)}(P/\Phi(P)) = P/\Phi(P)$, and hence

$$G/P \leqslant \mathrm{Aut}(P/\Phi(P)) \cong \mathrm{GL}_r(p).$$

A standard argument shows that G/P acts completely reducibly, otherwise $\mathrm{O}_p(G) > P$. [Recall that a module is called *completely reducible* if each submodule has a complementary submodule.]

(c) Suppose to the contrary that $G^{(i-1)}/G^{(i)}$ and $G^{(i)}/G^{(i+1)}$ are both (nontrivial) cyclic groups. Then $\mathrm{Aut}(G^{(i)}/G^{(i+1)})$ is abelian and so $C_G(G^{(i)}/G^{(i+1)}) \leqslant G'$. This implies that $G^{(i-1)}/G^{(i+1)}$ is abelian (being a cyclic extension of a central subgroup). This is a contradiction.

(d) To simplify notation assume that $G^{(i+1)} = 1$, and set $M = G^{(i-1)}$ and $N = G^{(i)}$. Since M/N is a minimal normal subgroup of G/N, it is elementary abelian. Since it is also cyclic, it has prime order, say p. If p divides $|N|$, then $M' = [M, N] < N$, a contradiction. Thus N has order coprime to p. Now $M \leqslant C_G(M) < N$ because M is abelian and the chief factor N/M does not centralize M. Therefore, $C_G(M) = M$ and G/M is a subgroup of $\mathrm{Aut}(M)$. Since $N = [M, N] \times C_N(M)$, it follows that $C_N(M) = 1$, or that M/N acts fixed-point-freely on N. By the Frattini argument, G is a split extension of M by $N_G(K)$ where K is Sylow-p subgroup of M.

(e) Since $G^{(i-1)}/G^{(i)}$ centralizes $G^{(i)}/G^{(i+1)}$, it follows that $G^{(i-1)}/G^{(i)}$ is not cyclic. Thus there exist primes q and p such that $G^{(i)}/G^{(i+1)} \cong C_q$

and $G^{(i-1)}/G^{(i)} \cong C_p \times C_p$. If $p \neq q$, then $G^{(i-1)}/G^{(i+1)}$ is abelian, a contradiction. Therefore $G^{(i-1)}/G^{(i+1)}$ is extraspecial of order p^3. $\square$

Notation. Let G have solvable length d. Write $n(G) = (n_1, n_2, \dots, n_d)$ where n_i is the composition length of the abelian group $G^{(i-1)}/G^{(i)}$. Note that $c(G) = n_1 + n_2 + \cdots + n_d$. The invariant $n(G)$ will provide a useful tool for classifying elements of $\mathrm{C}_{\mathbf{S}}(d)$. Let $K \ltimes N$ and $K \cdot N$ denote a split extension, and a not necessarily split extension, of N by K respectively. Let p, q, r, s denote primes. Let C_p and E_p denote cyclic groups, and extraspecial groups of order p and p^3 respectively. Denote the metacyclic group $\langle a, b \mid a^p = b^q = 1, b^a = b^k\rangle$ of order pq by $M_{p,q}$, where the order of k modulo q is p. Note that $q \equiv 1 \mod p$ and the isomorphism type of $M_{p,q}$ is independent of k. Let H denote an extension of the quaternion group Q_8 of order 8 by the symmetric group S_3, that has solvable length 4. There are two such groups, namely $\mathrm{GL}_2(3)$ and the binary octahedral group $\mathrm{BO} = \langle a, b, c \mid a^2 = b^3 = c^4 = abc\rangle$. Furthermore, $\mathrm{GL}_2(3) = S_3 \ltimes Q_8$ is a split extension, and BO is a nonsplit extension.

Theorem 2. *Let $\mathrm{c}_{\mathbf{S}}(d)$ denote minimal composition length of a finite solvable group with solvable length d. The values of $\mathrm{c}_{\mathbf{S}}(d)$, and the structure of $G \in \mathrm{C}_{\mathbf{S}}(d)$ for $d \leqslant 6$, are given below.*

d	0	1	2	3	4	5	6
$\mathrm{c}_{\mathbf{S}}(d)$	0	1	2	4	5	7	8
G	1	C_p	$M_{p,q}$	$M_{p,q} \ltimes C_r^2$	$M_{p,q} \ltimes E_r$	$H \ltimes C_s^2$	$H \cdot E_s$
				$C_p \ltimes E_r$	BO	$\mathrm{Sp}_2(3) \cdot E_s$	

Proof. Let $G \in \mathrm{C}_{\mathbf{S}}(d)$. If $d \leqslant 2$, then the structure of G is clear, and hence so too are the values of $\mathrm{c}_{\mathbf{S}}(d)$. Suppose now that $d \geqslant 3$. It follows from Lemma 1(c) that $n_i + n_{i+1} \geqslant 3$ for $i \geqslant 2$. Hence the possible values of $n(G)$ are as follows:

d	3	4	5	6
$n(G)$	$(1,1,2)$	$(1,1,2,1)$	$(1,1,2,1,2)$	$(1,1,2,1,2,1)$
	$(1,2,1)$		$(1,2,1,2,1)$	

The question arises as to whether each of the 6 above values of $n(G)$ arise for particular groups G. The answer is affirmative. There is a subgroup of the automorphism group of an exponent-p extraspecial group of order p^{2k+1} isomorphic to the general symplectic group $\mathrm{GSp}_{2k}(p)$, see [19, 7]. Thus we may form the split extension $\mathrm{GSp}_{2k}(p) \ltimes p^{2k+1}$. When $k = 1$ and $p = 3$ this group is $G = \mathrm{GL}_2(3) \ltimes E_3$ as $\mathrm{GSp}_2(3) \cong \mathrm{GL}_2(3)$. Now G has solvable length 6, and the quotients $G^{(i-1)}/G^{(i)}$ are C_2, C_3, $C_2 \times C_2$, C_2, $C_3 \times C_3$, C_3. Thus $n(G)$ equals $(1, 1, 2, 1, 2, 1)$. By taking quotients of G or G' we see that each of the 6 above values of $n(G)$ arise.

We shall now be more specific about the structure of an arbitrary group G such that $n(G)$ is one of the 6 above values. It is clear that $G \in \mathrm{C}_{\mathbf{S}}(d)$. If $n(G) = (1,1,2)$, then $G^{(2)}$ is not cyclic by Lemma 1(c), so $G^{(2)} \cong C_r^2$ for some prime r. By Lemma 1(d), G is a split extension $M_{p,q} \ltimes C_r^2$. Indeed, $M_{p,q} \leqslant \mathrm{GL}_2(r)$ acts irreducibly. If $n(G) = (1,2,1)$, then $G' = E_r$ is extraspecial of order r^3 by Lemma 1(e), and $G = C_p \ltimes E_r$ where C_p acts fixed-point-freely on $E_r/\Phi(E_r)$. When $n(G) = (1,1,2,1)$, then $G = M_{p,q} \cdot E_r$. Since $p \mid (q-1)$, $q \neq r$ and $pq \mid (r^2-1)(r^2-r)$, it follows that $G = M_{p,q} \ltimes E_r$ is split, unless $p = q-1 = r$ and $G = \mathrm{BO}$. Suppose that $n(G) = (1,1,2,1,2)$. Then $G^{(4)}$ is noncyclic, say C_s^2 where s is prime. Now $G^{(2)}/G^{(4)}$ is an extraspecial group by Lemma 1(e), and it acts irreducibly on $G^{(4)}$. This forces $G^{(2)}/G^{(4)}$ to be isomorphic to the quaternion group Q_8, or the dihedral group D_8, of order 8. As $\mathrm{Out}(D_8) \cong C_2$, and $\mathrm{Out}(Q_8) \cong S_3$, it follows that $H = G/G^{(4)}$ is an extension of Q_8 by S_3. Therefore, $H \cong \mathrm{GL}_2(3)$ or BO. By Lemma 1(d), G is a split extension $H \ltimes C_s^2$. The action of H on C_s^2 is irreducible, and exists only for certain odd primes s. Arguing as above, the structure of G satisfying $n(G) = (1,2,1,2,1)$ is $\mathrm{Sp}_2(3) \cdot E_s$, where $\mathrm{Sp}_2(3)$ denotes the symplectic group and $\mathrm{Sp}_2(3) \cong H'$. If $s \neq 3$, then $G = \mathrm{Sp}_2(3) \ltimes E_s$ is split, and if $s = 3$ then there exist nonisomorphic nonsplit extensions of E_s by $\mathrm{Sp}_2(3)$, see [6]. Finally when $n(G) = (1,1,2,1,2,1)$, $H = G/G^{(4)} \cong \mathrm{GL}_2(3)$ or BO and $G^{(4)} \cong E_s$ is extraspecial of order s^3. If $s = 3$, then $H \cong \mathrm{GL}_2(3)$, and if $s \neq 3$, then $H \cdot E_s$ is a split extension. □

3. The case $d = 7$

Before proving that $\mathrm{c}_{\mathbf{S}}(7) = 13$ in Theorem 7, we need four preliminary lemmas.

Lemma 3. *Let* $\mathrm{cr}(n)$ *denote the maximal solvable length of a completely reducible solvable subgroup of* $\mathrm{GL}_n(\mathbb{F})$*, where* $\mathbb{F}$ *is any field. Then*

n	1	2	3	4	5	6	7	8
$\mathrm{cr}(n)$	1	4	5	5	5	6	6	8

Proof. See [13] for an explicit formula for the function $\mathrm{cr}(n)$. □

Lemma 4. *Let P be a finite abelian group, and let Q be a solvable subgroup of* $\mathrm{Aut}(P)$ *with solvable length d.*

(a) *Then*

$\lvert P\rvert$	p	p^2	p^3	p^4
d	$\leqslant 1$	$\leqslant 4$	$\leqslant 5$	$\leqslant 6$

.

(b) *A subgroup chain $P = P_0 > P_1 > \cdots > P_n = 1$ is called* maximal *if $|P| = p^n$, or equivalently $|P_{i-1} : P_i| = p$ for $i = 1, \ldots, n$. If P is an abelian group of order dividing p^4, and Q stabilizes a maximal subgroup chain, then $d \leqslant 3$.*

Proof. (a) If P is elementary abelian of order p^n, then the maximum value of d is given in [13], Theorem A. In particular, $d = 1, 4, 5, 6$ when $n = 1, 2, 3, 4$. If P is not elementary abelian, then $P^p = \{g^p \mid g \in P\}$ is a proper nontrivial characteristic subgroup. Furthermore, the automorphisms of P centralizing both P/P^p and P^p, form an abelian group. The above table follows from these two facts.

(b) This is true when P is elementary abelian, as then Q is a subgroup of the upper triangular matrices. If P is not elementary abelian, then consider the groups P/P^p and P^p as above. □

Much more is known about primitive maximal solvable linear groups than is given in the following lemma, however, this simplified form is all that we require.

Lemma 5. *Let M be an absolutely irreducible primitive maximal solvable subgroup of $\mathrm{GL}_r(\mathbb{F})$ where $\mathbb{F}$ is a finite field. Then $Z := Z(M)$ is cyclic of order $|\mathbb{F}| - 1$. If F is the Fitting radical of M (the maximal nilpotent normal subgroup of M), then F/Z is elementary abelian of order r^2. If $r = p_1^{k_1} \cdots p_s^{k_s}$ where the p_i are distinct primes, then there exist extraspecial subgroups E_i of F of order $p_i^{2k_i+1}$ such that F is a central product $(E_1 \times \cdots \times E_s) \curlyvee Z$, and F is conjugate in $\mathrm{GL}_r(\mathbb{F})$ to $(E_1 \otimes \cdots \otimes E_s)Z$.*

Proof. The first two sentences follow from [17], Lemma 19.1 and Theorem 20.9, and the last sentence can be deduced from results on pages 141–146. A more convenient reference is [16], Theorems 2.5.13 and 2.5.19. □

The following result is proved in [4, 14].

Lemma 6. *Let $p \geqslant 3$ be a prime, and let P be a p-group satisfying $|P'/P''| = p^3$ and $P'' \neq 1$. Then*

$$P' = \gamma_2(P) > \gamma_3(P) > \gamma_4(P) > \gamma_5(P) = P''.$$

Theorem 7. *A finite solvable group with solvable length 7 has composition length at least 13, and this bound is best possible. More succinctly, $\mathrm{c}_{\mathbf{S}}(7) = 13$.*

Proof. As usual, our proof has two parts: (1) show that if $d(G) = 7$, then $c(G) \geqslant 13$, and (2) exhibit a group G with $d(G) = 7$ and $c(G) = 13$. The second part is deferred to Proposition 8 below.

Suppose that $d(G) = 7$. By the proof of Lemma 1(a), we may reduce to the case that G has a unique minimal normal subgroup. By Lemma 1(b), there is a (unique) prime p such that $P := \mathrm{O}_p(G)$ is nontrivial, and $Q := G/P$ is a completely reducible subgroup of $\mathrm{GL}_r(p)$ where $|P/\Phi(P)| = p^r$. If $d(P) \geqslant 4$, then $c(P) \geqslant 13$ by [4, 3], and hence $c(G) \geqslant 14$. If $d(P) = 1$, then $d(Q) \geqslant 6$ and $c(Q) \geqslant \mathrm{c}_{\mathbf{S}}(6) = 8$ by Theorem 2. However, $r \geqslant 6$ by Lemma 3, and so

$$c(G) = c(Q) + c(P) \geqslant 8 + 6 = 14.$$

The two remaining cases when $d(P) = 2$ or 3 require more detailed analyses.

Case (a) $d(P) = 2$. Now $d(Q) \geqslant 5$, so $c(Q) \geqslant \mathrm{c}_{\mathbf{S}}(5) = 7$ by Theorem 2. If $c(P) \geqslant 6$, then $c(G) = c(Q) + c(P) \geqslant 7 + 6 = 13$. Thus it suffices to consider the cases when $c(P) < 6$. Let $|P| = p^{r+s}$ where $|\Phi(P)| = p^s$. Then $r \geqslant 3$ by Lemma 3 and there are three cases when $c(P) < 6$, namely

$$(r, s) = (3, 1), (3, 2) \quad \text{and} \quad (4, 1).$$

We show that the first possibility never arises, and if the second or third arise, then $c(G) \geqslant 13$.

Subcase $(r, s) = (3, 1)$. In this case $\Phi(P) = P'$ has order p. If $Z(P) = P'$, then P is an extraspecial group with even composition length, a contradiction. Hence $\Phi(P) < Z(P) < P$ and since Q acts completely reducibly, $Q \leqslant \mathrm{GL}_1(p) \times \mathrm{GL}_2(p)$ by Lemma 1(b). Thus $d(Q) \leqslant 4$ by Lemma 3. This is a contradiction as $d(Q) \geqslant 5$. Hence this case never arises.

Subcase $(r, s) = (3, 2)$. Arguing as in the previous case, we see that $Q \leqslant \mathrm{GL}_3(p)$ is an irreducible subgroup. If Q does not act absolutely irreducibly, then $Q \leqslant \mathrm{GL}_1(p^3)$, and $d(Q) \leqslant 1$, a contradiction. If $Q \leqslant \mathrm{GL}_3(p)$ is an imprimitive subgroup, then $Q \leqslant \mathrm{GL}_1(p) \,\mathrm{wr}\, S_3$ and $d(Q) \leqslant 3$, a contradiction. In summary, $Q \leqslant \mathrm{GL}_3(p)$ acts absolutely irreducibly and primitively. Thus Q is a subgroup of an absolutely irreducible primitive maximal solvable subgroup M of $\mathrm{GL}_3(p)$. By Lemma 5 there are characteristic subgroups $Z \leqslant F \leqslant M$ such that F/Z is elementary abelian of order 3^2, $M/F \leqslant \mathrm{Sp}_2(3)$, and F' has order 3. Since $c(Q) \geqslant 7$ and $c(P) = 5$, we must eliminate the case when $c(G) = 12$. In this case, $c(Q) = 7$, $M/F \cong \mathrm{Sp}_2(3)$, and F contains an extraspecial subgroup of order 3^3 and exponent 3, and $M = QZ$. Since

$Z(Q) \leqslant Z(M)$, and M acts absolutely irreducibly, there is an element $z \in Z(Q)$ of order 3 which induces the scalar transformation $\omega 1$ on $P/\Phi(P)$ where ω is primitive cube root of 1 modulo p. We view z as an element of $G^{(4)}$ of order 3.

We show that $\Phi(P) \leqslant Z(P)$. If $\Phi(P) \nleqslant Z(P)$, then

$$\Phi(P) < Z(P)\Phi(P) < P.$$

This contradicts the fact that Q acts irreducibly on $P/\Phi(P)$. In summary, we know that $\Phi(P) \leqslant Z(P)$, and $g^z\Phi(P) = g^\omega\Phi(P)$ for all $g \in P$. Therefore,

$$[g_1, g_2]^z = [g_1^\omega, g_2^\omega] = [g_1, g_2]^{\omega^2} \qquad (g_2, g_2 \in P).$$

This proves that z acts nontrivially on P', and hence $\mathrm{Aut}(P')$ contains a subgroup with solvable length at least 5, contrary to Lemma 4(a). Thus we have proved $c(G) \geqslant 13$ in this case.

SUBCASE $(r, s) = (4, 1)$. Then $\Phi(P) = P'$ has order p, so $P' \leqslant Z(P)$. Since $d(P) > 1$, it follows that $p^2 \leqslant |P : Z(P)| \leqslant p^4$. If $|P : Z(P)| = p^2$, then it follows from Lemma 1(b) that $Q \leqslant \mathrm{GL}_2(p) \times \mathrm{GL}_2(p)$, and hence $d(Q) \leqslant 4$ by Lemma 3. This is a contradiction as $d(Q) \geqslant 5$. If $|P : Z(P)| = p^3$, then similar reasoning shows $Q \leqslant \mathrm{GL}_3(p) \times \mathrm{GL}_1(p)$. Arguing as in the previous subcase, Q acts absolutely irreducibly and primitively on $P/Z(P)$. Appealing as above to Lemma 5, a nontrivial element $z \in Z(Q)$ maps generators a_1, a_2, a_3, a_4 for P to $a_1^\omega, a_2^\omega, a_3^\omega, a_4$ modulo $\Phi(P)$ where ω is a primitive cube root of unity modulo p. Since $P' = \Phi(P) \leqslant Z(P)$, there is a well defined action of z on P'. Since $[a_i, a_j]^z = [a_i^z, a_j^z] = [a_i, a_j]^\omega$ or $[a_i, a_j]^{\omega^2}$, $z \in G^{(4)}$ acts nontrivially on P'. Thus $\mathrm{Aut}(P')$ contains a subgroup with solvable length at least 5, contrary to Lemma 4. Thus $c(G) \geqslant 13$ in this case also.

CASE (B) $d(P) = 3$. Since $P'' \neq 1$, $|P'/P''| \geqslant p^3$ by Hilfsatz 7.10 of [10]. If $p = 2$, then $\mathrm{GL}_2(2)$ and $\mathrm{GL}_3(2)$ are too small to accommodate a solvable subgroup Q with $d(Q) \geqslant 4$ and $c(Q) \geqslant c_{\mathbf{S}}(4) = 5$. Hence if $p = 2$, then $r \geqslant 4$ and

$$c(P) \geqslant c(P/\Phi(P)) + c(P'/P'') + c(P'') \geqslant 4 + 3 + 1 = 8.$$

Therefore $c(G) = c(Q) + c(P) \geqslant 5 + 8 = 13$ as desired. Assume now that $p \geqslant 3$ and $|P'/P''| = p^3$. By Lemma 6,

$$P' = \gamma_2(P) > \gamma_3(P) > \gamma_4(P) > \gamma_5(P) = P''.$$

Now P/P' acts nontrivially on P'/P''. Since $G^{(3)} \nleqslant P$, it follows that $\mathrm{Aut}(P'/P'')$ contains a subgroup with solvable length at least 4. This contradicts Lemma 4(b). Henceforth assume that $|P'/P''| \geqslant p^4$.

In summary, $c(Q) \geqslant 5$ and $c(P) \geqslant 7$, so $c(G) \geqslant 12$. Assume by way of contradiction that $c(G) = 12$. Then $c(Q) = 5$ and $c(P) = 7$. Since $d(Q) = 4$, we have $Q \in \mathrm{C}_{\mathbf{S}}(4)$. Thus $Q \cong \mathrm{GL}_2(3)$ or BO by Theorem 2. In addition, $|P/P'| = p^2$, $|P'/P''| = p^4$ and $|P''| = p$. Thus $\gamma_2(P)/\gamma_3(P)$ is cyclic, and so

$$P'' = [\gamma_2(P), \gamma_2(P)] = [\gamma_2(P), \gamma_3(P)] \leqslant \gamma_5(P).$$

If $P'' < \gamma_5(P)$, then $P' = \gamma_2(P) > \gamma_3(P) > \gamma_4(P) > \gamma_5(P) > P''$. However, P/P' acts nontrivially on the abelian group P'/P'' of order p^4. As Q acts irreducibly on P/P', it follows that $G^{(4)} = P$. Thus $\mathrm{Aut}(P'/P'')$ contains a subgroup with solvable length at least 4, contrary to Lemma 4(b). Hence $P'' = \gamma_5(P)$.

If the cyclic group $\gamma_2(P)/\gamma_3(P)$ has order at least p^2, then its order is exactly p^2, and we have the characteristic series

$$P' = \gamma_2(P) > \gamma_2(P)^p\gamma_3(P) > \gamma_3(P) > \gamma_4(P) > \gamma_5(P) = P''.$$

As above, this is impossible. Thus $|\gamma_2(P)/\gamma_3(P)| = p$, and $|\gamma_3(P)| = p^4$. Now $\gamma_3(P)$ is abelian as $[\gamma_3(P), \gamma_3(P)] \leqslant \gamma_6(P) = 1$. Exactly one of $|\gamma_3(P) : \gamma_4(P)|$ or $|\gamma_4(P) : \gamma_5(P)|$ has order p^2. Suppose that $\gamma_3(P)$ has a characteristic subgroup N of index p^2, and K is a solvable group of automorphisms of $\gamma_3(P)$. By Lemma 4(a), $K^{(4)}$ centralizes both $\gamma_3(P)/N$ and N. Since N is abelian, it follows that $K^{(5)} = 1$. However, $P'/\gamma_3(P)$ acts nontrivially on $\gamma_3(P)$ and $G^{(4)} \not\leqslant P'$, so $\mathrm{Aut}(\gamma_3(P))$ contains a subgroup with solvable length at least 6. This contradicts the fact that $K^{(5)} = 1$, and proves that $|\gamma_4(P) : \gamma_5(P)| = p^2$. Now $K = G/\gamma_3(P)$ satisfies $d(K) = 6$ and $c(K) = 8$. Thus $K \in \mathrm{C}_{\mathbf{S}}(6)$. By Theorem 2, $K \cong H \cdot E_p$ where $H \cong \mathrm{GL}_2(3)$ or BO.

Consider the section $G^{(3)}/\gamma_4(P)$. Since $G^{(4)} = P$ we have

$$|G^{(3)} : P| = 2, |P : P'| = p^2 \quad \text{and} \quad |P' : \gamma_3(P)| = |\gamma_3(P) : \gamma_4(P)| = p.$$

Let $z \in G^{(3)}$ have order 2. It follows from the structure of $G/\gamma_3(P)$ that z acts as the scalar transformation $-I$ on $P/P' \cong C_p \times C_p$. As p is odd, and z centralizes both $P'/\gamma_3(P)$ and $\gamma_3(P)/\gamma_4(P)$, it centralizes the abelian group $P'/\gamma_4(P)$. Let $g \in P$ and $h \in P'$. Then

$$[g, h] \equiv [g, h]^z \equiv [g^z, h^z] \equiv [g^{-1}, h] \equiv [g, h]^{-1} \bmod \gamma_4(P).$$

As p is odd and $[g, h]^2 \equiv 1 \mod \gamma_4(P)$, we see that $[P, P'] \subseteq \gamma_4(P)$. This is a contradiction as $\gamma_3(P) \not\subseteq \gamma_4(P)$.

In summary, we have proved in each case that if $d(G) = 7$, then $c(G) \geqslant 13$. □

The last case in Theorem 7 was difficult to eliminate. We can show that $\gamma_3(P)$ is either $(C_p)^4$ or $C_{p^2} \times (C_p)^2$. In either case, there exist a subgroup $H \ltimes E_p$ of $\mathrm{Aut}(\gamma_3(P))$ with solvable length 6 normalizing a subgroup chain $\gamma_3(P) = P_0 > P_1 > P_2 > P_3 = 1$ with $|P_0 : P_1| = |P_2 : P_3| = p$, and $|P_1 : P_2| = p^2$. Our contradiction was therefore subtle. It arose not because the action of $G/\gamma_3(P)$ on $\gamma_3(P)$ was untenable, rather because there was no extension of $\gamma_3(P)$ by $H \ltimes E_p$ having solvable length 7.

Proposition 8. *There exists a solvable group with solvable length* 7 *and composition length* 13. *Thus* $\mathrm{c}_{\mathbf{S}}(7) \leqslant 13$.

Proof. Let V be an r-dimensional vector space over a field $\mathbb{F}$. The homogeneous component $\Lambda^i V$ of the exterior algebra $\oplus_{i=0}^r \Lambda^i V$ has dimension $\binom{r}{i}$. Set $P = V \times \Lambda^2 V$, and define a binary operation on P via the rule

$$(v_1, w_1)(v_2, w_2) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2)$$

where $v_1, v_2 \in V, w_1, w_2 \in \Lambda^2 V$. Then P is a group. If $\mathrm{char}(\mathbb{F}) \neq 2$, then the derived subgroup P' equals $\{0\} \times \Lambda^2 V$ because $v_2 \wedge v_1 \neq -v_1 \wedge v_2$. The right action of $\mathrm{GL}_r(\mathbb{F})$ on P defined by $(v, w)g = (vg, w(g \wedge g))$ gives rise to a split extension $GL_r(\mathbb{F}) \ltimes P$. We are interested in the subgroup $K \ltimes P$ of this group when $r = 3$, $|\mathbb{F}| = p$ is an odd prime and $K \leqslant \mathrm{GL}_3(p)$ is isomorphic to $\mathrm{Sp}_2(3) \ltimes E_3$. If $p \equiv 1 \mod 3$, then there are faithful representations $\mathrm{Sp}_2(3) \ltimes E_3 \to \mathrm{GL}_3(p)$. [Indeed, when $p \equiv 1 \mod 9$, then there are faithful representations of the nonsplit extensions $\mathrm{Sp}_2(3) \cdot E_3 \to \mathrm{GL}_3(p)$.] Let $G = K \ltimes P$. Then $c(K) = 7$ and $c(P) = \binom{3}{1} + \binom{3}{2} = 6$, so $c(G) = c(K) + c(P) = 13$. We show now that $d(G) = 7$. An element $z \in K^{(4)}$ of order 3 induces the scalar transformation $\omega 1$ on $P/\Phi(P) \cong V$, where ω has order 3 modulo p. If $k \in K$ has matrix A relative to a basis e_1, e_2, e_3 for V, then $k \wedge k$ has matrix $\det(A)(A^{-1})^T$ relative to the basis $e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2$ for $\Lambda^2 V$. Therefore, z acts like $\omega^2 1$ on $\Phi(P) = P'$. This shows that $G^{(5)} = P$, and hence that $d(G) = 7$. □

4. The case $d = 8$

Theorem 9. *A finite solvable group with solvable length* 8 *has composition length at least* 15, *and this bound is best possible. More succinctly,* $\mathrm{c}_{\mathbf{S}}(8) = 15$.

Proof. As remarked in the Introduction, the group $\mathrm{GL}_2(3) \ltimes E_3 \ltimes 2^{6+1}$ of order $2^{11}3^4$ has solvable length 8. This proves that $\mathrm{c}_{\mathbf{S}}(8) \leqslant 15$. Since $\mathrm{c}_{\mathbf{S}}(7) = 13$, we see that $\mathrm{c}_{\mathbf{S}}(8) = 14$ or 15. We eliminate the case $\mathrm{c}_{\mathbf{S}}(8) = 14$.

Let $G \in \mathrm{C}_{\mathbf{S}}(8)$. Suppose that $P = \mathrm{O}_p(G)$ is nontrivial and $|P/\Phi(P)|$ equals p^r. Then $Q = G/P$ is a completely reducible subgroup of $\mathrm{GL}_r(p)$. If $d(P) \geqslant 4$, then $c(P) \geqslant 13$ by [4, 3], and hence $c(G) > 15$. If $d(P) = 1$, then $d(Q) \geqslant 7$ and $r \geqslant 8$ by Lemma 3. By Theorem 7, $c(Q) \geqslant 13$ so $c(G) \geqslant 13 + 8 = 21$. We shall now consider the two remaining cases: $d(P) = 2$ or 3.

Case $d(P) = 2$. Now $d(Q) \geqslant 6$, so $c(Q) \geqslant \mathrm{c}_{\mathbf{S}}(6) = 8$. By Lemma 3, $r \geqslant 6$ therefore $c(P) \geqslant 7$, and so $c(G) \geqslant 8 + 7 = 15$.

Case $d(P) = 3$. Now $d(Q) \geqslant 5$, so $c(Q) \geqslant \mathrm{c}_{\mathbf{S}}(5) = 7$. By Lemma 3, $r \geqslant 3$. Since $|P'/P''| \geqslant p^3$, it follows that $|P| \geqslant p^7$. Therefore $c(G) \geqslant 7 + 7 = 14$. Suppose that $c(G) = 14$. Then $c(Q) = 7$, $|P| = p^7$, $P' = \Phi(P)$, $|P' : P''| = p^3$ and $|P''| = p$. By Lemma 3, $Q \leqslant \mathrm{GL}_3(p)$ acts irreducibly. Arguing as in Theorem 2, Q acts absolutely irreducibly and primitively. Therefore, $G^{(5)} = P$. It follows from Lemma 5 that $Q \cong \mathrm{Sp}_2(3) \cdot E_3$ where E_3 has exponent 3. Now P/P' acts nontrivially on P'/P''. Therefore $\mathrm{Aut}(P'/P'')$ contains a subgroup with solvable length at least 6. This is impossible by Lemma 4(b). Hence $\mathrm{c}_{\mathbf{S}}(8) = 15$ as claimed. □

With more precise arguments, we can show that if $d(G) = 8$ and $c(G) = 15$, then $d(P) = 2$. By Lemma 3, G/P acts irreducibly on $P/\Phi(P)$, and so $\Phi(P) = P' = Z(P)$. Thus P is an extraspecial group of order p^{6+1} (and exponent p, if p is odd). Since $c(Q) = 8$ and $d(Q) = 6$, $Q \cong H \cdot E_s$ by Theorem 2. The representation theory of extraspecial groups shows that $s = 3$, and hence $Q \cong \mathrm{GL}_2(3) \ltimes E_3$. In addition, $p \equiv -1 \mod 3$, and Q' acts irreducibly but not absolutely irreducibly on $P/\Phi(P)$. In summary, elements of $\mathrm{C}_{\mathbf{S}}(8)$ have the form $(\mathrm{GL}_2(3) \ltimes E_3) \cdot p^{6+1}$.

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Department of Mathematics
Central Washington University
WA 98926-7424, USA
GlasbyS@cwu.edu