Modules induced from a normal subgroup of prime index

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Abstract. Let G be a finite group and H a normal subgroup of prime index p. Let V be an irreducible $\mathbb{F}H$-module and U a quotient of the induced $\mathbb{F}G$-module $V\!\uparrow$. We describe the structure of U, which is semisimple when $\mathrm{char}(\mathbb{F}) \neq p$ and uniserial if $\mathrm{char}(\mathbb{F}) = p$. Furthermore, we describe the division rings arising as endomorphism algebras of the simple components of U. We use techniques from noncommutative ring theory to study $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ and relate the right ideal structure of $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ to the submodule structure of $V\!\uparrow$.

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1. Introduction

Throughout this paper G will denote a finite group and H will denote a normal subgroup of prime index p. Furthermore, V will denote an irreducible (right) $\mathbb{F}H$-module, and $V\!\uparrow = V \otimes_{\mathbb{F}H} \mathbb{F}G$ is the associated induced $\mathbb{F}G$-module. Let a be an element of G not in H, and let $\Delta := \mathrm{End}_{\mathbb{F}H}(V)$ and $\Gamma := \mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$.

This paper is motivated by the following problem: "Given an irreducible $\mathbb{F}H$-module V, where $\mathbb{F}$ is an arbitrary field, and a quotient U of $V\!\uparrow$, determine the submodule structure of U and the endomorphism algebras of the simple modules." By Schur's lemma, Δ is a division algebra over $\mathbb{F}$, so we shall need techniques from noncommutative ring theory.

We determine the submodule structure of U by explicitly realizing $\mathrm{End}_{\mathbb{F}G}(U)$ as a direct sum of minimal right ideals, or as a local ring. It suffices to solve our problem in the case when $U = V\!\uparrow$. Henceforth $U = V\!\uparrow$.

In the case when $\mathbb{F}$ is algebraically closed of characteristic zero, it is well known that two cases arise. Either V is not G-stable and $V\!\uparrow$ is irreducible, or V is G-stable and $V\!\uparrow$ is a direct sum of p pairwise nonisomorphic irreducible submodules. In [GK96] the structure of $V\!\uparrow$ is analyzed in the case when $\mathbb{F}$ is an arbitrary field satisfying $\mathrm{char}(\mathbb{F}) \neq 0$. The assumption that $\mathrm{char}(\mathbb{F}) \neq 0$ was made to ensure that Δ is a field. The main theorem of [GK96] states that the structure of $V\!\uparrow$ is divided into five cases when V is G-stable. In this paper, we drop the hypothesis that $\mathrm{char}(\mathbb{F}) \neq 0$, and even more cases arise in the stable case (Theorems 5, 8 and 9). Fortunately, all these cases can be unified by considering the factorization of a certain binomial $t^p - \lambda$ in a twisted polynomial ring $\Delta[t;\alpha]$, which is a (left and right) principal ideal domain.

As we will focus on the case when $\mathbb{F}$ need not be algebraically closed, a crucial role will be played by the endomorphism algebra $\Delta = \mathrm{End}_{\mathbb{F}H}(V)$. In [GK96] the submodules of $V\!\uparrow$ are described *up to isomorphism*. As this paper is motivated by computational applications we will strive towards a higher standard: an explicit description of the vectors in the submodule, and an explicit description of the matrices in the endomorphism algebra of the submodule. This is easily achieved in the non-stable case, which we describe for the sake of completeness.

2. The non-stable case

Let $e_0, e_1, \ldots, e_{d-1}$ be an $\mathbb{F}$-basis for V and let $\sigma\colon H \to \mathrm{GL}(V)$ be the representation afforded by the irreducible $\mathbb{F}H$-module V relative to this basis. The *g-conjugate* of σ ($g \in G$) is the representation $g \mapsto (ghg^{-1})\sigma$, and we say that σ is *G-stable* if for each $g \in G$, σ is equivalent to its g-conjugate. In this section we shall assume that σ is *not* G-stable.

Let $\sigma\!\uparrow\colon G \to \mathrm{GL}(V\!\uparrow)$ be the representation afforded by $V\!\uparrow$ relative to the basis

$$e_0, \ldots, e_{d-1}, e_0 a, \ldots, e_{d-1} a, \ldots, e_0 a^{p-1}, \ldots, e_{d-1} a^{p-1}.$$

Note that $G/H = \langle aH \rangle$ has order p, and we are writing $e_i a^j$ rather than $e_i \otimes a^j$. Then

$$a\sigma\!\uparrow = \begin{pmatrix} 0 & I & & 0 \\ & & \ddots & \\ 0 & 0 & & I \\ a^p\sigma & 0 & & 0 \end{pmatrix}, \qquad h\sigma\!\uparrow = \begin{pmatrix} h\sigma & & & \\ & ({}^{a}h)\sigma & & \\ & & \ddots & \\ & & & ({}^{a^{p-1}}h)\sigma \end{pmatrix}$$

where $h \in H$ and ${}^{a^i}h = a^i h a^{-i}$. The elements of $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ are the matrices commuting with $G\sigma\!\uparrow$, namely the $p \times p$ block scalar matrices $\mathrm{diag}(\delta, \ldots, \delta)$ where $\delta \in \Delta$.

We shall henceforth assume that V is G-stable. In particular, assume that we know $\alpha \in \mathrm{Aut}_{\mathbb{F}}(V)$ satisfying

$$(1) \qquad (aha^{-1})\sigma = \alpha(h\sigma)\alpha^{-1} \qquad \text{for all } h \in H.$$

There are 'practical' methods for computing α. A crude method involves viewing $(ah_i a^{-1})\sigma\alpha = \alpha(h_i\sigma)$, where $H = \langle h_1, \ldots, h_r \rangle$, as a system of $(d/e)^2r$ homogeneous linear equations over Δ in $(d/e)^2$ unknowns where $e = |\Delta : \mathbb{F}|$. The solution space is 1-dimensional if V is G-stable, and 0-dimensional otherwise. A more sophisticated method, especially when $\mathrm{char}(\mathbb{F}) \neq 0$, involves using the meat-axe algorithm, see [P84], [HR94], [IL00], and [NP95]. There is also a recursive method for finding α which we shall not discuss here.

3. The elements of $\mathrm{End}_{\mathbb{F}G}(V{\uparrow})$

In this section we explicitly describe the matrices in $\Gamma = \mathrm{End}_{\mathbb{F}G}(V{\uparrow})$ and give an isomorphism $\Gamma \to (\Delta, \alpha, \lambda)$ where $(\Delta, \alpha, \lambda)$ is a general cyclic $\mathbb{F}$-algebra, see [L91, 14.5]. It is worth recalling that if $\mathrm{char}(\mathbb{F}) \neq 0$, then the endomorphism algebra Δ is commutative. Even in this case, though, Γ can be noncommutative.

Lemma 1. *If $i \in \mathbb{Z}$, then $\alpha^{-i}a^i\colon V \to Va^i$ is an $\mathbb{F}H$-isomorphism between the submodules V and Va^i of $V{\uparrow}{\downarrow}$.*

Proof. It follows from Eqn (1) that

$$va^iha^{-i} = v\alpha^ih\alpha^{-i} \qquad (i \in \mathbb{Z},\, v \in V)$$

Replacing v by $v\alpha^{-i}$ gives $v\alpha^{-i}a^ih = vh\alpha^{-i}a^i$. Hence $\alpha^{-i}a^i$ is an $\mathbb{F}H$-homomorphism, and since it is invertible, it is an $\mathbb{F}H$-isomorphism. □

Conjugation by α induces an automorphism of Δ, which we also call α. [Proof: Conjugating the equation $h\delta = \delta h$ by α and using (1) shows that $\alpha^{-1}\delta\alpha \in \Delta$.] We abbreviate $\alpha^{-1}\delta\alpha = \delta^\alpha$ by $\alpha(\delta)$. The reader can determine from the context whether the symbol α refers to an element of $\mathrm{Aut}_{\mathbb{F}}(V)$, or $\mathrm{Aut}_{\mathbb{F}}(\Delta)$.

It follows from Eqn (1) that

$$(a^pha^{-p})\sigma = \alpha^ph\sigma\alpha^{-p}$$

for all $h \in H$. Hence $\alpha^{-p}(a^p\sigma)$ centralizes H. Therefore $\lambda := \alpha^{-p}(a^p\sigma)$ lies in $\Delta^\times$. Setting $h = a^p$ in Eqn (1) shows $a^p\sigma = \alpha(a^p\sigma)\alpha^{-1}$. Thus

$$\alpha(\lambda) = \lambda^\alpha = (\alpha^{-p}(a^p\sigma))^\alpha = \alpha^{-p}(a^p\sigma) = \lambda.$$

Conjugating by $\alpha^p = (a^p\sigma)\lambda^{-1}$ induces an inner automorphism:

$$\alpha^p(\delta) = \delta^{\alpha^p} = \delta^{(a^p\sigma)\lambda^{-1}} = \delta^{\lambda^{-1}} = \lambda\delta\lambda^{-1} \qquad (\delta \in \Delta).$$

In summary, we have proved

Lemma 2. *The element $\alpha \in \mathrm{Aut}_{\mathbb{F}}V$ satisfying* Eqn (1) *induces via conjugation an automorphism of $\Delta = \mathrm{End}_{\mathbb{F}H}(V)$, also called α. There exists $\lambda \in \Delta^\times$ satisfying*

$$\alpha^{-p}(a^p\sigma) = \lambda, \quad \alpha(\lambda) = \lambda, \quad \alpha^p(\delta) = \lambda\delta\lambda^{-1} \qquad \text{(2a,b,c)}$$

for all $\delta \in \Delta$.

Theorem 3. *The representation* $\sigma\!\uparrow\colon G \to \mathrm{GL}(V\!\uparrow)$ *afforded by* $V\!\uparrow$ *relative to the* $\mathbb{F}$*-basis*

$$\begin{aligned} &e_0, e_1, \ldots, e_{d-1}, \ldots, e_0\alpha^{-i}a^i, e_1\alpha^{-i}a^i, \ldots, e_{d-1}\alpha^{-i}a^i, \\ &\ldots, e_0\alpha^{-(p-1)}a^{p-1}, e_1\alpha^{-(p-1)}a^{p-1}, \ldots, e_{d-1}\alpha^{-(p-1)}a^{p-1} \end{aligned} \tag{3}$$

for $V\!\uparrow$ *is given by*

$$a\sigma\!\uparrow = \begin{pmatrix} 0 & \alpha & & 0 \\ & & \ddots & \\ 0 & 0 & & \alpha \\ \alpha\lambda & 0 & & 0 \end{pmatrix}, \quad h\sigma\!\uparrow = \begin{pmatrix} h\sigma & & & \\ & h\sigma & & \\ & & \ddots & \\ & & & h\sigma \end{pmatrix} \tag{4a,b}$$

where $h \in H$. *Moreover, there is an isomorphism from the general cyclic algebra* $(\Delta, \alpha, \lambda)$ *to* $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$

$$(\Delta, \alpha, \lambda) \to \mathrm{End}_{\mathbb{F}G}(V\!\uparrow)\colon \sum_{i=0}^{p-1} \delta_i x^i \mapsto \sum_{i=0}^{p-1} D(\delta_i) X^i$$

where

$$X = \begin{pmatrix} 0 & I & & 0 \\ & & \ddots & \\ 0 & 0 & & I \\ \lambda & 0 & & 0 \end{pmatrix}, \quad D(\delta) = \begin{pmatrix} \delta & & & \\ & \alpha(\delta) & & \\ & & \ddots & \\ & & & \alpha^{p-1}(\delta) \end{pmatrix} \tag{5a,b}$$

and $\delta \in \Delta$.

Proof. By Lemma 1, $\alpha^{-i}a^i\colon V \to Va^i$ is an $\mathbb{F}H$-isomorphism. Hence $h\sigma\!\uparrow$ is the $p \times p$ block scalar matrix given by Eqn (4b). Similarly, $a\sigma\!\uparrow$ is given by Eqn (4a) as

$$(v\alpha^{-i}a^i)a = v\alpha\alpha^{-(i+1)}a^{i+1} \quad \text{and} \quad (v\alpha^{-(p-1)}a^{p-1})a = v\alpha(\alpha^{-p}a^p) = v\alpha\lambda$$

where the last step follows from Eqn (2a).

We follow [J96] and write $R = \Delta[t;\alpha]$ for the twisted polynomial ring with the usual addition, and multiplication determined by $t\delta = \alpha(\delta)t$ for $\delta \in \Delta$. The right ideal $(t^p - \lambda)R$ is two-sided as

$$t(t^p - \lambda) = (t^p - \lambda)t \quad \text{and} \quad \delta(t^p - \lambda) = (t^p - \lambda)\lambda^{-1}\delta\lambda$$

by virtue of Eqns (2b) and (2c). The general cyclic algebra $(\Delta, \alpha, \lambda)$ is defined to be the quotient ring $R/(t^p - \lambda)R$. Since R is a (left) euclidean domain, the elements of $(\Delta, \alpha, \lambda)$ may be written uniquely as $\sum_{i=0}^{p-1} \delta_i x^i$ where $x = t + (t^p - \lambda)R$, and multiplication is determined by the rules $x^p = \lambda$ and $x\delta = \alpha(\delta)x$ where $\delta \in \Delta$.

The matrices commuting with $H\sigma\!\uparrow$ are precisely the block matrices $(\delta_{i,j})_{0\le i,j<p}$ where $\delta_{i,j}\in\Delta$. To compute $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$, we determine the matrices $(\delta_{i,j})$ that commute with $a\sigma\!\uparrow$. If $0<i<p-1$, then comparing row $i-1$ of both sides of $(a\sigma\!\uparrow)(\delta_{i,j})=(\delta_{i,j})(a\sigma\!\uparrow)$ shows how to express $\delta_{i,j}$ in terms of the $\delta_{i-1,k}$. Similarly, row $p-1$ shows how to express $\delta_{p-1,j}$ in terms of $\delta_{0,k}$. It follows that a matrix $(\delta_{i,j})$ commuting with $a\sigma\!\uparrow$ is completely determined once we know the 0th row $\delta_{0,j}$. We show that the 0th row can be arbitrary. The 0th row of $\sum_{i=0}^{p-1}D(\delta_i)X^i$ is $(\delta_0,\delta_1,\dots,\delta_{p-1})$. This element lies in Γ as we show that $D(\delta_i),X\in\Gamma$.

To see that $D(\delta)\in\Gamma$, we show that $(a\sigma\!\uparrow)D(\delta)=D(\delta)(a\sigma\!\uparrow)$. The first product equals

$$(a\sigma\!\uparrow)D(\delta)=\begin{pmatrix}0 & \delta\alpha & & 0\\ & & \ddots & \\ 0 & 0 & & \alpha^{-(p-2)}\delta\alpha^{p-1}\\ \alpha\lambda\delta & 0 & & 0\end{pmatrix}$$

and the second product is identical if $\alpha\lambda\delta=\delta^{\alpha^{p-1}}\alpha\lambda$. However, this is true by Eqn (2c). To see that $X\in\Gamma$, write $a\sigma\!\uparrow=AX$ where $A=\mathrm{diag}(\alpha,\dots,\alpha)$. It follows from Eqn (2b) that A and X commute. Therefore $a\sigma\!\uparrow=AX$ and X commute.

In summary, elements of Γ may be written uniquely as $\sum_{i=0}^{p-1}D(\delta_i)X^i$ where $\delta_i\in\Delta$. Since $X^p=\lambda I$ and $XD(\delta)=D(\alpha(\delta))X$ it follows that the map $\sum_{i=0}^{p-1}\delta_i x^i\mapsto\sum_{i=0}^{p-1}D(\delta_i)X^i$ is an isomorphism $(\Delta,\alpha,\lambda)\to\Gamma$ as claimed. □

A consequence of Eqn (2c) is that α has order p or 1 modulo the inner automorphisms of Δ. It follows from the Skolem-Noether theorem [CR90,3.62] that the order of α modulo inner automorphisms is precisely the order of the restriction $\alpha|Z$, where $Z=Z(\Delta)$ is the centre of Δ.

4. The case when $\alpha|Z$ has order p

In this section we determine the structure of $\Gamma:=\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ in the case when α induces an automorphism of order p on the field $Z(\Delta)$.

Of primary interest to us is Part (a) of the following classical theorem. Although this result can be deduced from [J96, Theorem 1.1.22] and the fact that $t^p-\lambda$ is a 'two-sided maximal' element of $\Delta[t;\alpha]$, we prefer to give an elementary proof which generalizes [L91, Theorem 14.6].

Theorem 4. *Let Γ be the general cyclic algebra (Δ,α,λ) where $\lambda\neq 0$ and $\alpha(\lambda)=\lambda$. Suppose that $\alpha|Z(\Delta)$ has order p, and fixed subfield Z_0.*

Then
(a) Γ *is a simple* Z_0*-algebra,*
(b) $C_\Gamma(\Delta) = Z(\Delta)$,
(c) $Z(\Gamma) = Z_0$*, and*
(d) $|\Gamma : Z_0| = (p\mathrm{Deg}(\Delta))^2$ *where* $|\Delta : Z(\Delta)| = \mathrm{Deg}(\Delta)^2$.

Proof. The following proof does not assume that p is prime. Let $\gamma = \gamma_1 x^{i_1} + \cdots + \gamma_r x^{i_r}$ be a nonzero element of an ideal I of Γ, where $0 \le i_1 < \cdots < i_r < p$, $\gamma_i \in \Delta$, and r is chosen minimal. By minimality, each γ_i is nonzero. To prove Part (a) it suffices to prove that $r = 1$. Then $I = \Gamma$ as $\gamma_1 x^{i_1} \in I$ is a unit because γ_1 and x are both units. Assume now that $r > 1$. Then

$$(\gamma_1 \alpha^{i_1}(\delta)\gamma_1^{-1})\gamma - \gamma\delta = \sum_{k=2}^{r} \left(\gamma_1 \alpha^{i_1}(\delta)\gamma_1^{-1}\gamma_k - \gamma_k \alpha^{i_k}(\delta)\right) x^{i_k}$$

lies in I for each $\delta \in \Delta$. By the minimality of r, each coefficient of x^{i_k} is zero. This implies that α^{i_1} equals α^{i_k} modulo inner automorphisms for $k = 2, \ldots, r$. This contradiction proves Part (a).

The proofs of Parts (b) and (c) are straightforward, so we shall omit their proofs. Part (d) follows from $|\Gamma : \Delta| = |Z(\Delta) : Z_0| = p$, and $|\Delta : Z(\Delta)| = \mathrm{Deg}(\Delta)^2$ is a square. $\square$

Before proceeding to Theorem 5, we define the *left-* and *right-twisted powers*, $\mu^{\sqsupset i}$ and $\mu^{i \sqsubset}$, where $\mu \in \Delta$ and $i \in \mathbb{Z}$. These expressions are like norms, indeed Jacobson [J96] uses the notation $N_i(\mu)$ to suggest this. These "norms", however, are not multiplicative in general. Consider the twisted polynomial ring $\Delta[t;\alpha]$ and define

$$(\mu t)^i = \mu^{\sqsupset i} t^i, \quad \text{and} \quad (t\mu)^i = t^i \mu^{i \sqsubset}$$

for $\mu \in \Delta$ and $i \in \mathbb{Z}$. It follows from the power laws $(\mu t)^i (\mu t)^j = (\mu t)^{i+j}$ and $((\mu t)^i)^j = (\mu t)^{ij}$ that

$$\mu^{\sqsupset i} \alpha^i(\mu^{\sqsupset j}) = \mu^{\sqsupset (i+j)}, \quad \text{and} \quad \mu^{\sqsupset i} \alpha^i(\mu^{\sqsupset i}) \cdots \alpha^{i(j-1)}(\mu^{\sqsupset i}) = \mu^{\sqsupset (ij)}$$

for $i, j \in \mathbb{Z}$. Similar laws hold for right-twisted powers. The left-twisted powers of nonnegative integers can be defined by the recurrence relation

$$(6) \qquad \mu^{\sqsupset 0} = 1, \quad \text{and} \quad \mu^{\sqsupset (i+1)} = \mu^{\sqsupset i} \alpha^i(\mu) = \mu \alpha(\mu^{\sqsupset i}) \quad \text{for } i \ge 0,$$

and negative powers can be defined by $\mu^{\sqsupset -i} = \alpha^i(\mu^{\sqsupset i})^{-1}$.

It is important in the sequel whether or not λ^{-1} has a left-twisted pth root.

Theorem 5. *Let* V *be a* G*-stable irreducible* $\mathbb{F}H$*-module where* $H \lhd G$ *and* $|G/H| = p$ *is prime. Let* α *and* λ *be as in Lemma 2. Suppose that* $\alpha|Z$ *has order* p *where* $Z = Z(\Delta)$ *and* $\Delta = \mathrm{End}_{\mathbb{F}G}(V)$.

(a) *If the equation* $\mu^{\urcorner p} = \lambda^{-1}$ *has no solution for* $\mu \in \Delta^\times$, *then* $V\!\uparrow$ *is irreducible, and* $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ *is isomorphic to the general cyclic algebra* $(\Delta, \alpha, \lambda)$ *as per Theorem 3.*
(b) *If* $\mu \in \Delta^\times$ *satisfies* $\mu^{\urcorner p} = \lambda^{-1}$, *then* $V\!\uparrow = U(\mu_0) \dotplus \cdots \dotplus U(\mu_{p-1})$ *where*

$$U(\mu_j) = V \sum_{i=0}^{p-1} \mu_j^{\urcorner i} \alpha^{-i} a^i \qquad (j = 0, 1, \dots, p-1)$$

are isomorphic irreducible submodules satisfying $U(\mu_j){\downarrow} \cong V$, *and where* $\mu_j^{\urcorner p} = \lambda^{-1}$. *Moreover, if* $\rho\colon G \to \mathrm{GL}(U(\mu))$ *is the representation afforded by* $U(\mu)$ *relative to the basis* $e'_0, \dots, e'_{p-1}$ *where*

$$e'_j = e_j \sum_{i=0}^{p-1} \mu^{\urcorner i} \alpha^{-i} a^i \qquad (j = 0, 1, \dots, d-1),$$

then $a\rho = \alpha\mu^{-1}$, $h\rho = h\sigma$ *for* $h \in H$, *and*

$$\mathrm{End}_{\mathbb{F}G}(U(\mu)) = C_\Delta(\alpha\mu^{-1}) = \{\delta \in \Delta \mid \delta^\alpha = \delta^\mu\}.$$

Proof. By Theorem 4(a), $(\Delta, \alpha, \lambda)$ is a simple ring. In Part(a) more is true: $(\Delta, \alpha, \lambda)$ is a division ring by [J96, Theorem 1.3.16]. By Theorem 3, $(\Delta, \alpha, \lambda)$ is isomorphic to $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ and so we have proved that $V\!\uparrow$ is irreducible as desired.

Consider Part (b). Let $s = \mu t$ be an element of the twisted polynomial ring $\Delta[t;\alpha]$, then $s^i = (\mu t)^i = \mu^{\urcorner i} t^i$ and

$$s\delta = \mu t \delta = (\mu\alpha(\delta)\mu^{-1})\mu t = \mu\alpha(\delta)\mu^{-1} s.$$

Therefore the map $\Delta[t;\alpha] \to \Delta[s;\alpha\mu^{-1}]\colon \sum_{i=0}^{p-1} \delta_i t^i \mapsto \sum_{i=0}^{p-1} \delta_i (\mu^{\urcorner i})^{-1} s^i$ is an isomorphism. We are abusing notation here by identifying $\alpha\mu^{-1}$ in $\mathrm{Aut}_{\mathbb{F}}(V)$ with $\delta \mapsto \delta^{\alpha\mu^{-1}}$ in $\mathrm{Aut}_{\mathbb{F}}(\Delta)$. If $y = \mu x$, then

$$y^p = (\mu x)^p = \mu^{\urcorner p}\lambda = \lambda^{-1}\lambda = 1.$$

By taking quotients we get an isomorphism $(\Delta, \alpha, \lambda) \to (\Delta, \alpha\mu^{-1}, 1)$ given by

$$\sum_{i=0}^{p-1} \delta_i x^i \mapsto \sum_{i=0}^{p-1} \delta_i (\mu^{\urcorner i})^{-1} y^i$$

where $x = t + (t^p - \lambda)$ and $y = s + (s^p - 1)$.

As $y^p - 1 = (y-1)(y^{p-1} + \cdots + y + 1)$, and $\Delta[s, \alpha\mu^{-1}]$ is right euclidean it follows that $(y-1)(\Delta, \alpha\mu^{-1}, 1)$ is a maximal right ideal of $(\Delta, \alpha\mu^{-1}, 1)$. Now $y - 1$ corresponds to $\mu x - 1$ which corresponds to $D(\mu)X - 1$ whose kernel gives rise to the irreducible submodule $U(\mu)$ of $V\!\uparrow$ in the statement of Part(b). We shall reprove this, and prove a little more, using a more elementary argument.

Let U be a submodule of $V\!\uparrow$ satisfying $U\!\downarrow \cong V$. Let $\phi\colon V \to V\!\uparrow$ be an $\mathbb{F}H$-homomorphism such that $V\phi = U\!\downarrow$. Let $\pi_i\colon V\!\uparrow \to Va^i$ be the $\mathbb{F}H$-epimorphism given by $(\sum_{i=0}^{p-1} v_i\alpha^{-i}a^i)\pi_i = v_i\alpha^{-i}a^i$. Then $\delta_i = \phi\pi_i a^{-i}\alpha^i$ is an $\mathbb{F}H$-homomorphism $V \to V$, or an element of Δ. Since $\pi_0 + \pi_1 + \cdots + \pi_{p-1}$ is the identity map $1\colon V\!\uparrow \to V\!\uparrow$, it follows that

$$\phi = \phi 1 = \phi(\pi_0 + \pi_1 + \cdots + \pi_{p-1}) = \sum_{i=0}^{p-1} \delta_i \alpha^{-i} a^i.$$

We now view ϕ as a map $V \to U$ and note that $U = Ua$. Then $\alpha^{-1}a\colon V \to Va$, $a^{-1}\phi a\colon Va \to Ua$ and $\phi^{-1}\colon Ua \to V$ are each $\mathbb{F}H$-isomorphisms. Hence their composite, $(\alpha^{-1}a)(a^{-1}\phi a)\phi^{-1}$ is an isomorphism $V \to V$, denoted μ^{-1} where $\mu \in \Delta^\times$. Rearranging gives $\phi a = \alpha\mu^{-1}\phi$. Therefore,

$$(v\phi)a = \left(v\sum_{i=0}^{p-1} \delta_i\alpha^{-i}a^i\right)a = v\alpha\mu^{-1}\sum_{i=0}^{p-1} \delta_i\alpha^{-i}a^i$$

for all $v \in V$. The expression $(v\delta_i\alpha^{-i}a^i)a$ equals

$$v\delta_i\alpha\alpha^{-(i+1)}a^{i+1} = v\alpha\delta_i^\alpha\alpha^{-(i+1)}a^{i+1} = v\alpha\mu^{-1}\delta_{i+1}\alpha^{-(i+1)}a^{i+1}.$$

Setting $i = p-1$ gives

$$(v\delta_{p-1}\alpha^{-(p-1)}a^{p-1})a = v\alpha\delta_{p-1}^\alpha\alpha^{-p}a^p = v\alpha\delta_{p-1}^\alpha\lambda = v\alpha\mu^{-1}\delta_0.$$

Therefore $\delta_i^\alpha = \mu^{-1}\delta_{i+1}$ for $i = 0, \dots, p-2$ and $\delta_{p-1}^\alpha\lambda = \mu^{-1}\delta_0$. If $\delta_0 = 0$, then each $\delta_i = 0$ and $\phi = 0$, a contradiction. Thus $\delta_0 \neq 0$ and as $V\delta_0^{-1}\phi = U$, we may assume that $\delta_0 = 1$. It follows from Eqn (6) that $\delta_i = \mu^{\daleth i}$ is the solution to the recurrence relation: $\delta_0 = 1$ and $\mu\delta_i^\alpha = \delta_{i+1}$ for $i \geq 0$. Furthermore $\mu\delta_{p-1}^\alpha = \lambda^{-1}$ implies that $\mu^{\daleth p} = \lambda^{-1}$. In summary, any submodule U of $V\!\uparrow$ satisfying $U\!\downarrow \cong V$ equals $U(\mu)$ for some μ satisfying $\mu^{\daleth p} = \lambda^{-1}$. Furthermore, by retracing the above argument, if $\mu^{\daleth p} = \lambda^{-1}$, then $U(\mu)$ is an irreducible submodule of $V\!\uparrow$ satisfying $U\!\downarrow \cong V$.

As $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$ is a simple ring, $V\!\uparrow$ is a direct sum of isomorphic simple submodules. Therefore, $V\!\uparrow = U(\mu_0) \dotplus \cdots \dotplus U(\mu_{p-1})$ as desired. It follows from Lemma 1 that the representation $\rho\colon G \to \mathrm{GL}(V)$ satisfies $a\rho = \alpha\mu^{-1}$ and $h\rho = h\sigma$ for $h \in H$. Consequently, the matrices commuting with $G\rho$ equal the elements of Δ centralizing $a\rho$. Hence $\mathrm{End}_{\mathbb{F}G}(U(\mu)) = C_\Delta(\alpha\mu^{-1})$ as claimed. $\square$

5. The case when α is inner

In this section assume that $\alpha|Z(\Delta)$ has order 1, or equivalently by the Skolem-Noether theorem, that α is inner. Fix $\varepsilon \in \Delta^\times$ such that α is the inner automorphism $\alpha(\delta) = \varepsilon^{-1}\delta\varepsilon$. Clearly $\alpha(\varepsilon) = \varepsilon$ and by Eqn (2c) $\varepsilon^{-p}\delta\varepsilon^{p} = \alpha^{p}(\delta) = \lambda\delta\lambda^{-1}$. Therefore, $\eta = \varepsilon^{p}\lambda \in Z(\Delta)$. If $y = \varepsilon x$, then $y^{p} = \varepsilon^{\daleth p}x^{p} = \varepsilon^{p}\lambda = \eta$ and $y\delta = \varepsilon x\delta = \varepsilon\delta^{\varepsilon}x = \delta\varepsilon x = \delta y$. Hence

$$(\Delta, \alpha, \lambda) \to (\Delta, 1, \eta)\colon \sum_{i=0}^{p-1} \delta_i x^i \mapsto \sum_{i=0}^{p-1} \delta_i \varepsilon^{-i} y^i \tag{8}$$

is an isomorphism. Thus we may untwist $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$.

Theorem 6. *Let V be a G-stable irreducible $\mathbb{F}H$-module where $H \lhd G$ and $|G/H| = p$ is prime. Suppose that α induces the inner automorphism $\alpha(\delta) = \delta^{\varepsilon}$ of the division algebra $\Delta = \mathrm{End}_{\mathbb{F}H}(V)$. Then $\eta = \varepsilon^{p}\lambda \in Z^{\times}$ where $Z = Z(\Delta)$. Suppose that $s^{p} - \eta = \nu(s)\mu(s)$ where $\mu(s) = \sum_{i=0}^{m} \mu_i s^i$ and $\nu(s) = \sum_{i=0}^{p-m} \nu_i s^i$, are monic polynomials in $\Delta[s]$. Then $W_\mu = \sum_{i=0}^{m-1} V \sum_{j=0}^{p-m} \nu_j \varepsilon^{i+j}\alpha^{-(i+j)}a^{i+j}$ is a submodule of $V\!\uparrow$. Let $\rho\colon G \to \mathrm{GL}(W_\mu)$ be the representation afforded by W_μ relative to the basis*

$$e'_0, \dots, e'_{d-1}, \dots, e'_j(\varepsilon X)^k, \dots, e'_0(\varepsilon X)^{m-1}, \dots, e'_{d-1}(\varepsilon X)^{m-1} \tag{9}$$

where

$$e'_k = e_k \sum_{j=0}^{p-m} \nu_j \varepsilon^j \alpha^{-j} a^j = e_k \sum_{j=0}^{p-m} \nu_j (\varepsilon X)^j,$$

and X is given by Eqn (5a). *Then*

$$a\rho = \alpha\varepsilon^{-1} \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ -\mu_0 & -\mu_1 & & -\mu_{m-1} \end{pmatrix}, \tag{10}$$

and $h\rho = \mathrm{diag}(h\sigma, \dots, h\sigma)$ where $h \in H$. Moreover,

$$\mathrm{End}_{\mathbb{F}G}(W_\mu) = \left\{ \sum_{i=0}^{m-1} \delta_i (a\rho)^i \mid \delta_i \in \Delta \right\}.$$

If $\mu(s) \in Z[s]$, then $\mathrm{End}_{\mathbb{F}G}(W_\mu) \cong \Delta[s]/\mu(s)\Delta[s] \cong \Delta \otimes_Z \mathbb{K}$ where $\mathbb{K} = Z[s]/\mu(s)Z[s]$.

Proof. Arguing as in Theorem 5, we have a series of right ideals:

$$\nu(s)\Delta[s] \subseteq \Delta[s],\ \nu(y)(\Delta, 1, \eta) \subseteq (\Delta, 1, \eta),\ \nu(\varepsilon x)(\Delta, \alpha, \lambda) \subseteq (\Delta, \alpha, \lambda),$$

and $\sum_{i=0}^{n} D(\nu_i)(\varepsilon X)^i\Gamma$ is a right ideal of $\Gamma = \mathrm{End}_{\mathbb{F}G}(V\!\uparrow)$. This right ideal corresponds to the submodule $V\!\uparrow \sum_{i=0}^{n} D(\nu_i)(\varepsilon X)^i\Gamma$ of $V\!\uparrow$. It follows from Eqn (5a) and $(\varepsilon X)^p - \eta = 0$ that the minimum polynomial of εX equals $s^p - \eta$.

Let $v' = v\nu(\varepsilon X)$ where $v \in V$. Then

$$v'\mu(\varepsilon X) = v\nu(\varepsilon X)\mu(\varepsilon X) = v((\varepsilon X)^p - \eta) = v\,0 = 0. \tag{11}$$

This proves that (9) is a basis for

$$W_\mu = \mathrm{im}\;\nu(\varepsilon X) = \ker \nu(\varepsilon X) = \sum_{i=0}^{m-1} V \sum_{j=0}^{n} \nu_j \varepsilon^{i+j} \alpha^{-(i+j)} a^{i+j}$$

It follows from Lemma 1 that $h\rho = \mathrm{diag}(h\sigma, \ldots, h\sigma)$ is a block scalar matrix ($h \in H$). Since $a = \alpha X$,

$$v'(\varepsilon X)^i a = v'(\varepsilon X)^i \alpha X = v'\alpha\varepsilon^{-1}(\varepsilon X)^{i+1}. \tag{12}$$

It follows from Eqns (11) and (12) that the matrix for $a\rho$ is correct.

It is now a simple matter to show that $\left\{\sum_{i=0}^{m-1} \delta_i (a\rho)^i \mid \delta_i \in \Delta\right\}$ is contained in $\mathrm{End}_{\mathbb{F}G}(W_\mu)$. A familiar calculation shows that an element of $\mathrm{End}_{\mathbb{F}G}(W_\mu)$ is determined by the entries in its top row. As this may be arbitrary, we have found all the elements of $\mathrm{End}_{\mathbb{F}G}(W_\mu)$. $\square$

It follows from Theorem 6 that a necessary condition for W_μ to be irreducible is that $\mu(s)$ is irreducible in $\Delta[s]$. Lemma 7 describes an important case when $\mathrm{End}_{\mathbb{F}G}(W_\mu)$ is a division ring, and hence W_μ is irreducible. The following proof follows Prof. Deitmar's suggestion [D02].

Lemma 7. *Let Δ be a division algebra with center $\mathbb{F}$, and let $\mu(s) \in \mathbb{F}[s]$ be irreducible of prime degree. Suppose that no $\delta \in \Delta$ satisfies $\mu(\delta) = 0$. Then the quotient ring $\Delta[s]/\mu(s)\Delta[s]$ is a division algebra.*

Proof. Let $\mathbb{K} = \mathbb{F}[s]/\mu(s)\mathbb{F}[s]$. Then $\mathbb{K}$ is a field and $|\mathbb{K} : \mathbb{F}| = \deg \mu(s)$ is prime. Clearly $\mu(s)\Delta[s]$ is a two-sided ideal of $\Delta[s]$, and $\Delta[s]/\mu(s)\Delta[s]$ is isomorphic to $\Delta_\mathbb{K} = \Delta \otimes_\mathbb{F} \mathbb{K}$. By [L91, 15.1(3)], $\Delta_\mathbb{K}$ is a central simple $\mathbb{K}$-algebra, and hence is isomorphic to $M_n(D)$ for some division algebra D over $\mathbb{F}$. The *degree* of D and the *Schur index* of $\Delta_\mathbb{K}$ are defined as follows

$$\mathrm{Deg}(D) = (\dim_\mathbb{F} D)^{1/2} \quad \text{and} \quad \mathrm{Ind}(\Delta_\mathbb{K}) = \mathrm{Deg}(D).$$

By [P82, Prop. 13.4], $\mathrm{Ind}(\Delta_\mathbb{K})$ divides $\mathrm{Ind}(\Delta)$, and $\mathrm{Ind}(\Delta)$ divides $|\mathbb{K} : \mathbb{F}|\,\mathrm{Ind}(\Delta_\mathbb{K})$. Thus either

$$\mathrm{Ind}(\Delta_\mathbb{K}) = \mathrm{Ind}(\Delta) = \mathrm{Deg}(\Delta) = \mathrm{Deg}(D_\mathbb{K})$$

and $\Delta_\mathbb{K}$ is a division algebra by [P82, Prop. 13.4(ii)], or $\mathrm{Ind}(\Delta)$ equals $|\mathbb{K} : \mathbb{F}|\,\mathrm{Ind}(\Delta_\mathbb{K})$. If the second case occurred, then by [P82, Cor. 13.4], $\mathbb{K}$ is isomorphic to a subfield of Δ, and so $\mu(s)$ has a root in Δ, contrary to our hypothesis. $\square$

If $\eta \notin \Delta^p$, then $\eta \notin Z^p$ and so $s^p - \eta$ is irreducible in $Z[s]$, and it follows from Lemma 7 that $V{\uparrow} = W_{s^p-\eta}$ is irreducible. Note that $\mathrm{End}_{\mathbb{F}G}(V{\uparrow}) \cong \Delta \otimes Z[\eta^{1/p}]$ is a division algebra.

6. The case when α is inner and $\xi^p = \eta$

In this section we shall assume that $\xi \in \Delta^\times$ satisfies $\xi^p - \eta = 0$. Let $y = \varepsilon x$ and $z = \xi^{-1}y = \xi^{-1}\varepsilon x$. It is useful to consider the isomorphisms $(\Delta, \alpha, \lambda) \to (\Delta, 1, \eta) \to (\Delta, 1, 1)$ defined by $x \mapsto \varepsilon^{-1}y$ and $y \mapsto \xi z$. Note y and z are central in $(\Delta, 1, \eta)$ and $(\Delta, 1, 1)$ respectively, and $y^p = \eta$ and $z^p = 1$.

Theorem 8. *Let V be a G-stable irreducible $\mathbb{F}H$-module where $H \lhd G$ and $|G/H| = p$ is prime. Suppose that α induces the inner automorphism $\alpha(\delta) = \delta^\varepsilon$ of the division algebra $\Delta = \mathrm{End}_{\mathbb{F}H}(V)$. Set $\eta = \varepsilon^p\lambda$ and let $\xi, \omega \in \Delta$ satisfy $\xi^p = \eta$ and $\omega^p = 1$. Then $\xi \in Z = Z(\Delta)$.*
(a) *If $\mathrm{char}(\mathbb{F}) \neq p$ and $\omega \neq 1$, then $V{\uparrow}$ is an internal direct sum*

$$V{\uparrow} = U(\xi) \dotplus U(\xi\omega) \dotplus \cdots \dotplus U(\xi\omega^{p-1})$$

where

$$U(\xi\omega^j) = V \sum_{i=0}^{p-1} (\xi\omega^j)^{-i}\varepsilon^i\alpha^{-i}a^i$$

is irreducible, and $U(\xi\omega) \cong U(\xi\omega')$ if and only if ω and ω' are conjugate in Δ. If $\mu(s)$ is an irreducible factor of $s^p - \eta$ in $Z[s]$, then W_μ defined in Theorem 6 is a Wedderburn component of $V{\uparrow}$, and $W_\mu = U(\theta_1) \dotplus \cdots \dotplus U(\theta_n)$ where $\theta_1, \dots, \theta_n$ are the roots of $\mu(s)$ in the field $Z(\xi, \omega)$. In addition, the representation $\rho_\theta \colon G \to \mathrm{GL}(U(\theta))$ afforded by $U(\theta)$ relative to the basis $e'_0, \dots, e'_{d-1}$ where

$$e'_j = e_j \sum_{i=0}^{p-1} \theta^{-i}\varepsilon^i\alpha^{-i}a^i$$

satisfies

$$a\rho_\theta = \alpha\varepsilon^{-1}\theta \qquad \text{and} \qquad h\rho_\theta = h\sigma \tag{12a,b}$$

for $h \in H$, and $\mathrm{End}_{\mathbb{F}G}(U(\theta)) = C_\Delta(\theta)$.
(b) *If $\mathrm{char}(\mathbb{F}) = p$, then $\omega = 1$ and $V{\uparrow}$ is uniserial with unique composition series $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_p = V{\uparrow}$ where*

$$W_k = \sum_{i=1}^{k} V \sum_{j=0}^{p-i} \binom{i+j-1}{j} \xi^{-j}\varepsilon^j\alpha^{-j}a^j.$$

Moreover, $W_{k-1}/W_k \cong U(\xi)$ for $k = 1, \dots, p$ and $\mathrm{End}_{\mathbb{F}G}(U(\xi)) = \Delta$.

Proof. Since $z\delta = \delta z$, we see that $(\xi^{-1}\varepsilon x)\delta = \delta(\xi^{-1}\varepsilon x)$. This implies that $\xi^{-1}\delta = \delta\xi^{-1}$ and so $\xi \in Z$.
Case (a): Now $(\xi\omega)^p = \xi^p\omega^p = \eta$, hence

$$y^p - \eta = y^p - (\xi\omega)^p = (y - \xi\omega)\left(\sum_{i=0}^{p-1}(\xi\omega)^{p-1-i}y^i\right). \tag{13}$$

Therefore $V{\uparrow}\sum_{i=0}^{p-1}(\xi\omega)^{p-1-i}(\varepsilon X)^i\Gamma$ is a submodule of $V{\uparrow}$ where X is given by Eqn (5a). We show directly that $U(\xi\omega)$ is a submodule of $V{\uparrow}$. This follows from

$$\begin{aligned}(v(\xi\omega)^{-i}\varepsilon^i\alpha^{-i}a^i)a &= v\alpha(\alpha^{-1}(\xi\omega)^{-i}\varepsilon^i\alpha)\alpha^{-(i+1)}a^{i+1}\\ &= v\alpha\varepsilon^{-1}\xi\omega(\xi\omega)^{-(i+1)}\varepsilon^{i+1}\alpha^{-(i+1)}a^{i+1}\end{aligned} \tag{14}$$

and setting $i = p-1$ in the right-hand side of Eqn (14) gives

$$v\alpha\varepsilon^{-1}\xi\omega(\xi\omega)^{-p}\varepsilon^p\alpha^{-p}a^p = v\alpha\varepsilon^{-1}\xi\omega\eta^{-1}\varepsilon^p\lambda = v\alpha\varepsilon^{-1}\xi\omega.$$

As $U(\xi\omega){\downarrow} \cong V$, we see that $U(\xi\omega)$ is an irreducible $\mathbb{F}G$-submodule of $V{\uparrow}$. Setting $\theta = \xi\omega$ establishes the truth of Eqns (12a,b).

We may calculate $\mathrm{Hom}(U(\xi\omega), U(\xi\omega'))$ directly by finding all δ in $\mathrm{End}_{\mathbb{F}}(V)$ that intertwine $\rho_{\xi\omega}$ and $\rho_{\xi\omega'}$. As δ intertwines $h\rho_{\xi\omega}$ and $h\rho_{\xi\omega'}$, it follows that δ commutes with $H\sigma$, and hence $\delta \in \Delta$. Also

$$\delta(\alpha\varepsilon^{-1}\xi\omega) = (\alpha\varepsilon^{-1}\xi\omega')\delta$$

so $\delta^{\alpha\varepsilon^{-1}}\xi\omega = \xi\omega'\delta$. Since $\xi \in Z^\times$ and $\delta^{\alpha\varepsilon^{-1}} = \delta$, this amounts to $\delta\omega = \omega'\delta$. Setting $i = j$ shows that $\mathrm{End}_{\mathbb{F}G}(U(\xi\omega)) = C_\Delta(\omega)$.

The Galois group $\mathrm{Gal}(Z(\omega) : Z)$ is cyclic of order dividing $p-1$. Also ω and ω' are conjugate in $\mathrm{Gal}(Z(\omega) : Z)$ if and only if they share the same minimal polynomial over Z. The latter holds by Dixon's Theorem [L91, 16.8] if and only if ω and ω' are conjugate in Δ. Note that ω and ω' share the same minimal polynomial over Z precisely when $\xi\omega$ and $\xi\omega'$ share the same minimal polynomial. This proves that W_μ is a Wedderburn component of $V{\uparrow}$.

Case (b): Suppose now that $\mathrm{char}(\mathbb{F}) = p$. Then $\omega = 1$ and Eqn (13) becomes $y^p - \eta = (y-\xi)^p = (y-\xi)(\sum_{i=0}^{p-1}\binom{p-1}{i}(-\xi)^{p-1-i}y^i)$. As

$$\Gamma = \mathrm{End}_{\mathbb{F}G}(V{\uparrow}) \cong (\Delta, \alpha, \lambda) \cong (\Delta, 1, \eta) \cong (\Delta, 1, 1) \cong \Delta[z]/(z-1)^p\Delta[z]$$

has a unique composition series, so too does $V{\uparrow}$. By noting that $z = \xi^{-1}\varepsilon x$ and $D(\xi^{-1}\varepsilon) = \xi^{-1}\varepsilon$, we see that $W_i = V{\uparrow}(\xi^{-1}\varepsilon X - 1)^{p-i}\Gamma$ defines the unique composition series for $V{\uparrow}$ where X is given by Eqn (5a).

Let R be the diagonal matrix $\mathrm{diag}(1, \xi^{-1}\varepsilon, \ldots, (\xi^{-1}\varepsilon)^{p-1})$, and let S be the matrix whose (i,j)th block is the binomial coefficient $\binom{i}{j}$ where

$0 \le i,j < p$. A direct calculation verifies that $R(\xi^{-1}\varepsilon X)R^{-1} = C$ and $S^{-1}CS = J$ where

$$C = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ 1 & 0 & & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 1 & & \\ & \ddots & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

Therefore $\xi^{-1}\varepsilon X - 1 = T^{-1}(J-1)T$ where $T = S^{-1}R$, and hence

$$W_k = V{\uparrow}(\xi^{-1}\varepsilon X - 1)^{p-k} = V{\uparrow}\, T^{-1}(J-1)^{p-k}T = V{\uparrow}(J-1)^{p-k}T.$$

It is easily seen that $\mathrm{im}(J-1)^{p-k} = \ker(J-1)^k$ is the subspace $(0,\dots,0,V,\dots,V)$ where the first V is in column $p-k$. The (i,j)th entry of $T = S^{-1}R$ is $(-1)^{i+j}\binom{i}{j}(\xi^{-1}\varepsilon)^j$. The last row of T gives

$$W_1 = V\sum_{j=0}^{p-1}(-1)^{p-1+j}\binom{p-1}{j}(\xi^{-1}\varepsilon)^j\alpha^{-j}a^j.$$

More generally, the last k rows of T give

$$W_k = \sum_{i=1}^{k} V\sum_{j=0}^{p-i}(-1)^{p-i+j}\binom{p-i}{j}(\xi^{-1}\varepsilon)^j\alpha^{-j}a^j.$$

Since $p-i-\ell = -(i+\ell)$ in a field (such as $\mathbb{F}$) of characteristic p, we see that $\binom{p-i}{j} = (-1)^j\binom{i+j-1}{j}$ and the formula for W_k simplifies to

$$W_k = \sum_{i=1}^{k} V\sum_{j=0}^{p-i}\binom{i+j-1}{j}\xi^{-j}\varepsilon^j\alpha^{-j}a^j.$$

Setting $k=1$ shows $W_1 = U(\xi)$. A direct calculation shows that $W_{i-1}/W_i \cong U(\xi)$. We showed in Part (a) that $\mathrm{End}_{\mathbb{F}G}(U(\xi))$ equals $C_\Delta(\xi) = \Delta$. $\square$

In Case (a), $C_\Delta(\xi\omega)$ equals Δ precisely when $\omega \in Z$. If Δ is the rational quaternions, and ω is a primitive cube root of unity, then $C_\Delta(\omega)$ equals $\mathbb{Q}(\omega)$. There are infinitely many primitive cube roots of 1 in this case, and they form a conjugacy class of Δ by Dixon's Theorem (as they all satisfy the irreducible polynomial s^2+s+1 over $\mathbb{Q}$). Thus isomorphism of the submodules $U(\xi\omega)$ is governed by conjugacy in Δ, and not conjugacy in $\mathrm{Gal}(\mathbb{Q}(\omega):\mathbb{Q})$.

Finally, it remains to generalize Theorem 8(a) to allow for the possibility that Δ may not contain a primitive pth root of 1.

Theorem 9. *Let V be a G-stable irreducible $\mathbb{F}H$-module where $H \lhd G$ and $|G/H| = p$ is prime. Suppose that $\varepsilon,\xi \in \Delta$ satisfy $\alpha(\delta) = \delta^\varepsilon$*

($\delta \in \Delta$) and $\xi^p - \eta = 0$ where $\eta = \varepsilon^p \lambda \in Z = Z(\Delta)$. In addition, suppose that $\operatorname{char}(\mathbb{F}) \neq p$. *Then $V\!\uparrow$ is an internal direct sum*

$$V\!\uparrow = W_{\mu_1} \dotplus \cdots \dotplus W_{\mu_r}$$

where $s^p - \eta = \mu_1(s) \cdots \mu_r(s)$ is a factorization into monic irreducibles over Z, and where W_μ defined in Theorem 6. If $\mu(s)$ is a monic irreducible factor of $s^p - \eta$, and $\mu(s) = \nu_1(s) \cdots \nu_n(s)$ where the $\nu_i(s)$ are monic and irreducible in $\Delta[s]$, then W_μ is a Wedderburn component of $V\!\uparrow$, and $W_\mu \cong W_{\nu_n}^{\oplus n}$ where W_{ν_n} is an irreducible $\mathbb{F}G$-module and $\operatorname{End}_{\mathbb{F}G}(W_{\nu_n})$ *is given in Theorem 6. In addition,*

$$\operatorname{End}_{\mathbb{F}G}(W_{\nu_n}) \cong B/\nu_n(s)\Delta[s]$$

where $B = \{\delta(s) \in \Delta[s] \mid \delta(s)\nu_n(s) \in \nu_n(s)\Delta[s]\}$ is the idealizer of the right ideal $\nu_n(s)\Delta[s]$.

Proof. Since $\operatorname{char}(\mathbb{F}) \neq p$, the monic polynomials $\mu_1(s), \ldots, \mu_r(s)$ are distinct and pairwise coprime in $Z[s]$. From this it follows that $V\!\uparrow$ equals $W_{\mu_1} \dotplus \cdots \dotplus W_{\mu_r}$. By Theorem 6, $\operatorname{End}_{\mathbb{F}G}(W_\mu) \cong \Delta_{\mathbb{K}}$ where $\Delta_{\mathbb{K}} \cong \Delta[s]/\mu(s)\Delta[s] \cong \Delta \otimes_Z \mathbb{K}$, and $\mathbb{K}$ is the field $Z[s]/\mu(s)Z[s]$. By [L91, 15.1(3)], $\Delta_{\mathbb{K}}$ is a simple ring. Therefore $\mu(s)\Delta[s]$ is a two-sided maximal ideal of $\Delta[s]$, and so $\mu(s)$ is called a two-sided maximal element of $\Delta[s]$. By [J96, Theorem 1.2.19(b)], $\Delta_{\mathbb{K}} \cong M_n(D)$ where D is the division ring $B/\nu_n(s)\Delta[s]$. Moreover, $Z(\Delta_{\mathbb{K}}) \cong Z(M_n(D))$ so $\mathbb{K} \cong Z(D)$. Thus $W_\mu \cong W_{\nu_n}^{\oplus n}$ where W_{ν_n} is an irreducible submodule of $V\!\uparrow$ and $\operatorname{End}_{\mathbb{F}G}(W_{\nu_n}) \cong D$. In addition, $\nu_1, \ldots, \nu_n$ are similar [J96, Def. 1.2.7], and $W_{\nu_1}, \ldots, W_{\nu_n}$ are isomorphic.

If $\mu(s), \mu'(s)$ are distinct monic irreducible factors of $s^p - \eta$ in $Z[s]$ and $\nu(s), \nu'(s)$ in $\Delta[s]$ are monic irreducible factors of $\mu(s)$ and $\mu'(s)$ respectively, then it follows from [J96, Def. 1.2.7] that $\nu(s)$ and $\nu'(s)$ are not similar. This means that an irreducible summand of W_μ is not isomorphic to an irreducible summand of $W_{\mu'}$. Hence the W_μ are Wedderburn components as claimed. □

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