Irreducible modules and normal subgroups of prime index

S. P. Glasby

School of Mathematics and Statistics

University of Sydney, NSW 2006, Australia

and

L. G. Kovács

School of Mathematical Sciences

Australian National University, Canberra, ACT 0200, Australia

Abstract. Let $\mathbb{F}$ be a field, G a finite group, H a normal subgroup of prime index p, and V an irreducible $\mathbb{F}H$-module. If $\mathbb{F}$ is algebraically closed and of characteristic 0, the $\mathbb{F}G$-module induced from V is either irreducible or a direct sum of p pairwise nonisomorphic irreducible modules. It is shown here that if $\mathbb{F}$ is not assumed algebraically closed and its characteristic is not 0, then there are not two but six possibilities for the structure of the induced module.

1. Introduction

Throughout this paper, $\mathbb{F}$ denotes a field, G a finite group, and H a normal subgroup of prime index p in G.

Given a representation σ of H over $\mathbb{F}$ and an element g of G, the *g-conjugate* of σ is the composite of σ with the automorphism $h \mapsto ghg^{-1}$ of H. The G-conjugates are the g-conjugates with $g \in G$. If σ is equivalent to all

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its G-conjugates, we say that it is *G-stable*. Conjugacy and stability of $\mathbb{F}H$-modules are defined similarly. This paper is concerned with the structure of $\mathbb{F}G$-modules induced from irreducible $\mathbb{F}H$-modules. It is well known that if the $\mathbb{F}H$-module in question is not G-stable then the induced module is irreducible: therefore we restrict attention to the G-stable case. Specifically, throughout the paper, V denotes a G-stable irreducible $\mathbb{F}H$-module.

Recall that in the classical case ($\mathbb{F}$ algebraically closed of characteristic 0) the induced $\mathbb{F}G$-module (which we write simply as $V\uparrow$) is the direct sum $U_1 \oplus \cdots \oplus U_p$ of p pairwise nonisomorphic irreducibles, and the restrictions $U_i\downarrow$ are all isomorphic to V. The aim of this note is to explore, with computational applications in mind, what one can say about $V\uparrow$ when $\mathbb{F}$ is finite. A little extra argument will show that the result does not get any more complicated if one allows $\mathbb{F}$ to be any field of prime characteristic.

Since we are not assuming that $\mathbb{F}$ is algebraically closed, a key role will be played by the endomorphism ring of V: set $\mathrm{End}_{\mathbb{F}H} V = \mathbb{E}$. By Schur's Lemma, $\mathbb{E}$ is a division algebra over $\mathbb{F}$. Thus if $\mathbb{F}$ is finite then, by a theorem of Wedderburn, $\mathbb{E}$ is also a field. More generally, we shall see that if $\mathbb{F}$ is of prime characteristic then $\mathbb{E}$ is a field that is obtainable from $\mathbb{F}$ by adjoining a root of 1; in particular, $\mathbb{E}|\mathbb{F}$ is a Galois extension with cyclic Galois group.

Using the natural action of $\mathbb{E}$ on V, we may consider V as an $\mathbb{E}$-space rather than an $\mathbb{F}$-space. The $\mathbb{E}H$-module so obtained will be denoted by $V^{\mathbb{E}}$. One important question for the structure of $V\uparrow$ is whether $V^{\mathbb{E}}$ is also G-stable. If the answer is affirmative, we say that V is *absolutely G-stable*.

We need one more piece of notation before we can state our theorem. When the characteristic of $\mathbb{F}$ is not p, we write $\mathbb{E}\big(\sqrt[p]{1}\big)$ for the field obtained from $\mathbb{E}$ by adjoining a primitive pth root of 1, and s for the degree $|\mathbb{E}\big(\sqrt[p]{1}\big) : \mathbb{E}|$ of this field over $\mathbb{E}$, noting that s is a divisor of $p-1$.

THEOREM. *Let $\mathbb{F}$ be a field of prime characteristic, G a finite group, H a normal subgroup of prime index p, and V a G-stable irreducible $\mathbb{F}H$-module with $\mathbb{E} = \mathrm{End}_{\mathbb{F}H} V$.*

(a) *If V is not absolutely G-stable, then $V\uparrow$ is $U^{\oplus p}$, the direct sum of p copies of an irreducible $\mathbb{F}G$-module U. Here* $\mathrm{End}_{\mathbb{F}G} U$ *is a subfield of* $\mathbb{E}$ *that contains* $\mathbb{F}$ *and is such that* $|\mathbb{E} : \mathrm{End}_{\mathbb{F}G} U| = p$, *and* $U\downarrow = V$.

(b) *If V is absolutely G-stable and the characteristic of $\mathbb{F}$ is p, then $V\uparrow$ is a uniserial module of composition length p, and its composition factors are pairwise isomorphic. If U is one of these composition factors, then* $\mathrm{End}_{\mathbb{F}G} U = \mathbb{E}$ *and* $U\downarrow = V$.

(c) *If V is absolutely G-stable, if the characteristic of $\mathbb{F}$ is not p, and if $\mathbb{E}$ does not contain a primitive pth root of 1 (that is, if $s > 1$), then $V\uparrow = U \oplus W_1 \oplus \cdots \oplus W_{(p-1)/s}$ where U and the W_j are pairwise nonisomorphic irreducibles such that* $\mathrm{End}_{\mathbb{F}G} U = \mathbb{E}$ *and* $\mathrm{End}_{\mathbb{F}G} W_j = \mathbb{E}\big(\sqrt[p]{1}\big)$ *while* $U\downarrow = V$ *and* $W_j\downarrow = V^{\oplus s}$.

(d) *If V is absolutely G-stable, if the characteristic of $\mathbb{F}$ is not p, and if $\mathbb{E}$ does contain a primitive pth root of 1 (that is, if $s = 1$), then* either *$V\uparrow$ is the direct sum $U_1 \oplus \cdots \oplus U_p$ of p pairwise nonisomorphic irreducibles, each with* $\mathrm{End}_{\mathbb{F}G} U_i = \mathbb{E}$ *and* $U_i\downarrow = V$, or *$V\uparrow$ itself is irreducible. In the latter case,* $\mathrm{End}_{\mathbb{F}G}(V\uparrow)$ *is a degree p extension of $\mathbb{E}$ obtainable by adjoining a root of 1, and of course* $V\uparrow\downarrow = V^{\oplus p}$.

Case (a) cannot arise when G is abelian, but it does arise when H is cyclic: let q be any prime power, $\mathbb{F} = GF(q)$, $G = \Gamma L(1, q^p) \cap GL(p, q)$, $H = GL(1, q^p)$, and V the restriction of the natural $\mathbb{F}GL(p, q)$-module. All other cases arise even with finite $\mathbb{F}$ and cyclic G. In case (d), it is irrelevant whether $\mathbb{F}$ itself contains a primitive pth root of 1. Note that if V were not G-stable, $V\uparrow$ would be irreducible but with $\mathrm{End}_{\mathbb{F}G}(V\uparrow) = \mathbb{E}$ and $V\uparrow\downarrow \ncong V^{\oplus p}$: hence we do not consider that in that case the structure of $V\uparrow$ would be 'the same' as in the second part of case (d).

The methods we use hardly go beyond what can be found in Huppert and Blackburn [2] (see particularly Sections VII.1, VII.4 and VII.9), and much of the result itself may well be at least in the folklore: we have made no systematic attempt to track down references. We aim here for a coordinate-

free treatment, leaving algorithmic considerations for another paper. Where there is no danger of confusion, we often abuse language by not distinguishing between a module and its isomorphism class (or between isomorphism and equality of modules).

Implicit in the Theorem is an overview of the irreducible $\mathbb{F}G$-modules whose restriction to H is isomorphic to V. The proof of the Theorem starts with a careful analysis of these in the case when $\mathbb{F}$ is finite: this occupies Section 2. Changing from V to $V^{\mathbb{E}}$ is not the only way we need to change the field of scalars: Section 3 is devoted to a discussion of the relevant options. Sections 4, 5, 6 prove the Theorem under the assumption that $\mathbb{F}$ is finite. Finally, Section 7 extends the arguments to infinite $\mathbb{F}$ of prime characteristic, using a (presumably well-known) result to the effect that the theory of completely reducible representations of a finite group over such an $\mathbb{F}$ is always 'the same' as over a suitable finite subfield.

2. Extending to G a stable irreducible representation of H

Until the last section of the paper, we assume that $\mathbb{F}$ is finite. We know then that $\mathbb{E}$ is a field, and we refer to it as the *endomorphism field* of V.

In preparation for the proof of the Theorem, in this section we focus on a particular aspect: given a G-stable irreducible $\mathbb{F}H$-module V, what can be said about the existence and uniqueness of $\mathbb{F}G$-modules U such that $U{\downarrow} = V$? By Nakayama Reciprocity, such a U is both a submodule and a factormodule of $V{\uparrow}$; indeed, it is a direct summand of $V{\uparrow}$ when the characteristic is not p (see VII.9.4 in [2]).

It will be more convenient to conduct this discussion in terms of representations, adapting a familiar argument. Let

$$\sigma\colon H \to \mathrm{Aut}_{\mathbb{F}}\, V, \qquad h \mapsto h^{\sigma}$$

stand for the representation afforded by V. We think of $\mathbb{E}$ as the centralizer of H^{σ} in $\mathrm{End}_{\mathbb{F}}\, V$. The assumption that (the isomorphism type of) V is

G-stable means that to each element g of G there exist α_g in $\mathrm{Aut}_{\mathbb{F}} V$ such that

$$(ghg^{-1})^\sigma = \alpha_g h^\sigma \alpha_g^{-1} \qquad \text{for all } h \text{ in } H.$$

For a given g, the elements α_g of $\mathrm{Aut}_{\mathbb{F}} V$ with this property normalize the multiplicative group $\mathbb{E}^\times$ of nonzero elements of $\mathbb{E}$ and form a coset modulo $\mathbb{E}^\times$. Let N stand for the normalizer of H^σ in $\mathrm{Aut}_{\mathbb{F}} V$: it is easy to verify that $G \to N/\mathbb{E}^\times$, $g \mapsto \alpha_g \mathbb{E}^\times$ is a homomorphism. It follows that conjugation by $(\alpha_g)^p$ or α_{g^p} or by $(g^p)^\sigma$ gives the same automorphism of H^σ, whence

$$(\alpha_g)^{-p}(g^p)^\sigma \in \mathbb{E}^\times.$$

Choose an element a such that $G = \langle a \rangle H$, and write α_a simply as α. Note that $(a^p)^\sigma$ and α commute (put $g = a$ and $h = a^p$ in the second last displayed formula).

If there is a representation, ρ say, of G over $\mathbb{F}$ whose restriction to H is σ, then $a^\rho \in \alpha \mathbb{E}^\times$ and $(a^\rho)^p = (a^p)^\sigma$. Conversely, if there is an ε in $\mathbb{E}^\times$ such that $(\alpha\varepsilon)^p = (a^p)^\sigma$, then σ extends to a ρ such that $a^\rho = \alpha\varepsilon$ (see, for example, VII.9.8 in [2]). Thus the existence of U boils down to this: is there an ε in $\mathbb{E}^\times$ such that $(\alpha\varepsilon)^p = (a^p)^\sigma$?

Consider first the case that V is G-stable even as $\mathbb{E}H$-module: that is, α centralizes $\mathbb{E}^\times$. This is what we have called the *absolutely stable case.* What is wanted then is that ε be a pth root of $\alpha^{-p}(a^p)^\sigma$ in $\mathbb{E}^\times$. If $\mathbb{E}^\times$ has no element of order p (note this is always so when the characteristic of $\mathbb{F}$ is p, but can also happen when that characteristic is not p), then each element has precisely one pth root in $\mathbb{E}^\times$: so there is precisely one ρ which extends σ. On the other hand, if $\mathbb{E}^\times$ does have an element of order p, then $\alpha^{-p}(a^p)^\sigma$ has either p distinct roots in $\mathbb{E}$ or none at all. When there are p, let ε, ε' be two of them, and define two extensions of σ by setting $a^\rho = \alpha\varepsilon$ and $a^{\rho'} = \alpha\varepsilon'$. If these were equivalent, ρ' would be ρ followed by an inner automorphism of $\mathrm{Aut}_{\mathbb{F}} V$. Since on H they agree with σ, this inner automorphism would have to be induced by an element of $\mathbb{E}^\times$. However, in this case $\mathbb{E}$ centralizes

not only H^σ but a^ρ as well. Thus when there are p distinct ε, they yield p pairwise inequivalent ρ which extend σ.

Consider next the case that V is not G-stable as an $\mathbb{E}H$-module: that is, α does not centralize $\mathbb{E}^\times$. This is the *stable but not absolutely stable case.* Of course α still normalizes $\mathbb{E}^\times$; by conjugation, it induces a nontrivial field automorphism on $\mathbb{E}$, which has order p (because $\alpha^{-p}(a^p)^\sigma \in \mathbb{E}$ and $(a^p)^\sigma$ does centralize $\mathbb{E}$). The fixed points form a subfield, $\mathbb{E}_p$ say, which contains $\mathbb{F}$ and $\alpha^{-p}(a^p)^\sigma$, and is such that $|\mathbb{E} : \mathbb{E}_p| = p$. Denote the cardinality of $\mathbb{E}_p$ by q. There is an integer k such that $0 < k < p$ and

$$\alpha^{-1}\xi\alpha = \xi^{q^k} \qquad \text{for each } \xi \text{ in } \mathbb{E}.$$

Set $r = (q^p-1)/(q-1)$. Using that $\xi^{q^p} = \xi$, it is easy to see that $(\alpha\varepsilon)^p = \alpha^p\varepsilon^r$. Our question therefore comes to this: does $\alpha^{-p}(a^p)^\sigma$ have an rth root in $\mathbb{E}$? Since $\alpha^{-p}(a^p)^\sigma \in \mathbb{E}_p^\times$ and since r is the index of $\mathbb{E}_p^\times$ in the cyclic group $\mathbb{E}^\times$, the answer is always yes; indeed, our element has precisely r distinct rth roots in $\mathbb{E}^\times$. If ε is one of them, the others are the $\delta\varepsilon$ as δ ranges through the elements of the unique subgroup of order r in $\mathbb{E}^\times$. Define two extensions of σ by setting $a^\rho = \alpha\varepsilon$ and $a^{\rho'} = \alpha\delta\varepsilon$. Since r and $(q^k - 1)/(q - 1)$ are relatively prime and since the subgroup of order r in $\mathbb{E}^\times$ has index $q - 1$, it follows that $\delta = \eta^{q^k-1}$ for some η in $\mathbb{E}^\times$. One can readily verify (using the last displayed equation) that $a^{\rho'} = \eta a^\rho \eta^{-1}$, whence ρ' is equivalent to ρ. We conclude that while in this case there are r distinct choices for ε, there is only one equivalence class of extensions ρ of σ. (In Section 4, we shall rederive the results of this paragraph, and more, along different lines.)

We have now exhausted all possibilities. However, for later use we need to say a little more about the case when $\mathbb{E} = \mathbb{F}$ (that is, σ is absolutely irreducible) and there is no ρ that extends σ. Then $\mathbb{E}$ does contain primitive pth roots of 1 but $\alpha^{-p}(a^p)^\sigma$ has no pth root in $\mathbb{E}$, and therefore the polynomial $x^p - \alpha^{-p}(a^p)^\sigma$ is irreducible over $\mathbb{E}$. Let $\mathbb{E}^p$ denote the unique degree p extension of $\mathbb{E}$: this is the splitting field over $\mathbb{E}$ of our polynomial. Note that $\alpha^{-p}(a^p)^\sigma$ has p distinct pth roots ε in $\mathbb{E}^p$, and these roots are permuted

transitively by the Galois group $\mathrm{Gal}\big(\mathbb{E}^p\big|\mathbb{E}\big)$. Viewed as a representation over $\mathbb{E}^p$, σ is still irreducible and G-stable, and over $\mathbb{E}^p$ it has p extensions ρ that are pairwise inequivalent but $\mathrm{Gal}\big(\mathbb{E}^p\big|\mathbb{E}\big)$-conjugate.

While we have not proclaimed the conclusions of this discussion as formal propositions, we shall make use of them in the subsequent sections and incorporate them in the lemmas proved there.

3. Changing fields

In the second last paragraph, it proved expedient to view an $\mathbb{E}$-representation of H as an $\mathbb{E}^p$-representation. There are in fact three ways of changing fields that are important in our context, and it is as well to distinguish them carefully.

The first was mentioned already in the Introduction: one may take advantage of the fact that, by its very definition, $\mathbb{E}$ has an action on V and that action commutes with the action of H. We have agreed to denote by $V^{\mathbb{E}}$ the $\mathbb{E}H$-module obtained from V in this way. Note that $V^{\mathbb{E}}$ consists of the same elements as V, but we think of it as an $\mathbb{E}$-space rather than an $\mathbb{F}$-space. In particular, $\dim_{\mathbb{E}} V^{\mathbb{E}} = (\dim_{\mathbb{F}} V)/|\mathbb{E}:\mathbb{F}|$. It will be important for us that $V^{\mathbb{E}}$ is absolutely irreducible (for $\mathrm{End}_{\mathbb{E}H} V^{\mathbb{E}} = \mathbb{E}$).

Although we shall not use it here, we mention that a similar change of fields may be applied to direct sums of copies of V. The endomorphism ring of such a module is a full matrix algebra over $\mathbb{E}$, and the centre of that algebra is isomorphic to $\mathbb{E}$. This is so in a very strong sense: the projections of the direct sum to the direct summands yield specific isomorphisms from the centre of the matrix algebra to the endomorphism fields of those direct summands. It is therefore possible to consider such a direct sum an $\mathbb{E}H$-module in a choice-free way. Note that $(V^{\oplus k})^{\mathbb{E}} = (V^{\mathbb{E}})^{\oplus k}$.

Still more generally, one could change in this way to any subfield of $\mathbb{E}$ (containing $\mathbb{F}$), without necessarily going all the way to $\mathbb{E}$. However, that is as far as one can force generality: this kind of change only applies to modules

that are either irreducible or direct sums of isomorphic irreducibles, and only to subfields of the relevant endomorphism field. Given this limitation, it would be wasteful to appropriate the notation adopted here on anything more than a strictly temporary basis.

The second way to change fields is also obvious. It goes in the opposite direction: we change from a large field to a small one, forgetting the action of the rest of the large field. This move is not subject to limitations: when we view an $\mathbb{E}H$-module X as an $\mathbb{F}H$-module and denote the module so obtained by $X_{\mathbb{F}}$, $\mathbb{E}$ could be any field and $\mathbb{F}$ any subfield of it. The new module $X_{\mathbb{F}}$ consists of the same elements as the old one, the only difference is that there is less action on $X_{\mathbb{F}}$ than on X. Accordingly, $\dim_{\mathbb{F}} X_{\mathbb{F}} = (\dim_{\mathbb{E}} X)|\mathbb{E} : \mathbb{F}|$. This way of changing fields is discussed in VII.1.16 of [2], though the notation used there is different. If $\mathbb{E}$ is finite and $X_{\mathbb{F}}$ happens to be irreducible, then $\mathbb{E}$ is certainly contained in the endomorphism field of $X_{\mathbb{F}}$ so it makes sense to consider $(X_{\mathbb{F}})^{\mathbb{E}}$, and then of course $(X_{\mathbb{F}})^{\mathbb{E}} = X$. Similarly, $(V^{\mathbb{E}})_{\mathbb{F}} = V$.

The third change of fields makes an $\mathbb{E}H$-module from V by forming $V \otimes_{\mathbb{F}} \mathbb{E}$. Given the notation we have already chosen, we are not free to follow Huppert and Blackburn [2] in denoting this $\mathbb{E}H$-module by $V_{\mathbb{E}}$, or Curtis and Reiner [1] in denoting it by $V^{\mathbb{E}}$: we shall make do without that kind of shorthand. Note that $\dim_{\mathbb{E}}(V \otimes_{\mathbb{F}} \mathbb{E}) = \dim_{\mathbb{F}} V$, and $(V \otimes_{\mathbb{F}} \mathbb{E})_{\mathbb{F}} = V^{\oplus|\mathbb{E}:\mathbb{F}|}$. This move is also available for any $\mathbb{E}$, $\mathbb{F}$, V, but until the last section we shall only use it with $\mathbb{F}$ finite, V irreducible, and $\mathbb{E} = \mathrm{End}_{\mathbb{F}H} V$. In this case, $V \otimes_{\mathbb{F}} \mathbb{E}$ is a direct sum of $|\mathbb{E} : \mathbb{F}|$ pairwise nonisomorphic but $\mathrm{Gal}\big(\mathbb{E}\,\big|\,\mathbb{F}\big)$-conjugate, absolutely irreducible modules:

$$V \otimes_{\mathbb{F}} \mathbb{E} = V_1 \oplus \cdots \oplus V_{|\mathbb{E}:\mathbb{F}|}.$$

(In Chapter VII of [2], first combine 1.15 with 1.18b to justify this without the adjective 'absolutely', then use 1.12 and 1.4b to show that each $\mathrm{End}_{\mathbb{E}H} V_i$ is $\mathbb{E}$.) As $(V \otimes_{\mathbb{F}} \mathbb{E})_{\mathbb{F}} \cong V^{\oplus|\mathbb{E}:\mathbb{F}|}$, each $(V_i)_{\mathbb{F}}$ is isomorphic to V.

It is important to understand that although $\mathbb{E}$ has a natural action on V, in forming $V \otimes_{\mathbb{F}} \mathbb{E}$ only the action of the subfield $\mathbb{F}$ is used, the action of

the rest of $\mathbb{E}$ is ignored. The $\mathbb{E}H$-module $V^{\mathbb{E}}$ made from V by exploiting the natural action of $\mathbb{E}$ is isomorphic to one of the irreducible direct summands V_i of $V \otimes_{\mathbb{F}} \mathbb{E}$, but which summand that is depends on which $\mathbb{F}$-isomorphism we choose from the 'concrete' copy $\mathrm{End}_{\mathbb{F}H} V$ of $\mathbb{E}$ to the 'abstract' copy (with the natural action on V forgotten) that was used in forming the tensor product $V \otimes_{\mathbb{F}} \mathbb{E}$. Having chosen such an isomorphism, we are still free to choose the numbering of the V_i. It will be convenient to coordinate these choices so that $V_1 \cong V^{\mathbb{E}}$.

The G-stability of V obviously implies that $V \otimes_{\mathbb{F}} \mathbb{E}$ is G-stable, but of course it does not imply that $V^{\mathbb{E}}$, or any one of the V_i, is G-stable. However, all Galois conjugates of G-stable H-modules are obviously G-stable, so if one of the V_i is G-stable then so are all the others. What we called the *stable but not absolutely stable case* could therefore have been defined as that in which V is G-stable but at least one of the V_i is not (and so not one of the V_i is) G-stable. Similarly, the *absolutely stable case* is that in which at least one of the V_i is (and therefore all of the V_i are) G-stable.

4. The stable but not absolutely stable case

We are now ready to prove part (a) of the Theorem under the assumption that $\mathbb{F}$ is finite. The notation of the previous section will be retained; in particular, the V_i are numbered so that $V_1 \cong V^{\mathbb{E}}$. The assumption particular to part (a) is that V is G-stable but $V^{\mathbb{E}}$ and V_1 are not.

The first point to note is that, by Clifford's Theorem, the induced $\mathbb{E}G$-module $V_1{\uparrow}$ is irreducible. The second point is that the G-stability of $V \otimes_{\mathbb{F}} \mathbb{E}$ implies that conjugation action by G permutes (the isomorphism types of) the V_i; as V_1 is not G-stable, this action is nontrivial, so it follows that the kernel of the action is precisely H.

The Galois group $\mathrm{Gal}\big(\mathbb{E}\,\big|\,\mathbb{F}\big)$ permutes the set of the (isomorphism types of the) V_i regularly, and this action commutes with the action of G. Since a regular abelian group is its own centralizer in the ambient symmetric group,

it follows that G/H acts just like a subgroup of $\mathrm{Gal}\big(\mathbb{E}\big|\mathbb{F}\big)$. Since $\mathrm{Gal}\big(\mathbb{E}\big|\mathbb{F}\big)$ is cyclic, it has at most one subgroup of order p. We conclude that it does have one, and that G/H acts as that unique subgroup. Call that subgroup Γ, and let $\mathbb{E}_p$ be the subfield of $\mathbb{E}$ that consists of the elements fixed by Γ: this contains $\mathbb{F}$, and is the only subfield of $\mathbb{E}$ with the property that $|\mathbb{E} : \mathbb{E}_p| = p$.

Since a character induced from a normal subgroup vanishes outside that normal subgroup, the previous paragraph proves that the nonzero values of the character of G afforded by the irreducible $\mathbb{E}G$-module $V_1{\uparrow}$ are the same as the nonzero values of the sum of the Γ-conjugates of the character of H afforded by V_1. It follows that all values of the character of G afforded by $V_1{\uparrow}$ are fixed by Γ and so lie in $\mathbb{E}_p$. Since nothing more in the Galois group fixes (setwise) the set of isomorphism types of the G-conjugates of V_1, this set of character values cannot lie in any proper subfield of $\mathbb{E}_p$ that contains $\mathbb{F}$. That is, the character values (together with $\mathbb{F}$) generate $\mathbb{E}_p$. By a theorem of Brauer (VII.1.17 in [2]), there is an $\mathbb{E}_pG$-module X such that $X \otimes_{\mathbb{E}_p} \mathbb{E} \cong V_1{\uparrow}$. Set $U = X_{\mathbb{F}}$; by the last sentence of VII.1.16e in [2], this U is irreducible.

We know that $(V_1)_{\mathbb{F}} \cong V$, so $(V_1{\uparrow})_{\mathbb{F}} = (V_1)_{\mathbb{F}}{\uparrow} \cong V{\uparrow}$. On the other hand, $(V_1{\uparrow})_{\mathbb{E}_p} \cong X^{\oplus p}$, so $(V_1{\uparrow})_{\mathbb{F}} \cong U^{\oplus p}$. This proves that $V{\uparrow} \cong U^{\oplus p}$, and therefore the restriction $U{\downarrow}$ must be isomorphic to V. As we have seen before, any $\mathbb{F}G$-module whose restriction to H is V must be a submodule of $V{\uparrow}$ and therefore must be isomorphic to U. We have proved the first sentence of the following.

Lemma 1. *If $\mathbb{F}$ is finite and V is G-stable but $V^{\mathbb{E}}$ is not, then there is a unique isomorphism class of $\mathbb{F}G$-modules U such that $U{\downarrow} \cong V$, and in this case $V{\uparrow} \cong U^{\oplus p}$. Also, $|\mathbb{E} : \mathbb{F}|$ is divisible by p, and $\mathrm{End}_{\mathbb{F}G} U = \mathbb{E}_p$ where $\mathbb{E}_p$ is the unique subfield of $\mathbb{E}$ that contains $\mathbb{F}$ and is such that $|\mathbb{E} : \mathbb{E}_p| = p$.*

To prove the last claim, argue as follows: $\mathbb{E}_p$ is a maximal subfield of $\mathbb{E}$; it commutes with the action of G on U (by the definition of U in terms of X), nothing more than $\mathbb{E}$ can commute with the action of $\mathbb{F}H$ (according to the definition of $\mathbb{E}$), and $\mathbb{E}$ itself does not commute with the action of G (because

$V^{\mathbb{E}}$ is not G-stable).

The example mentioned in the Introduction shows that the case discussed here can occur regardless of whether the characteristic of $\mathbb{F}$ is the same as the index p, and can occur even when H is cyclic (and so G is metacyclic). It cannot occur if G is abelian, for then α must centralize H^σ and $\mathbb{E}$ is the $\mathbb{F}$-linear span of H^σ.

5. The absolutely stable case in characteristic p

LEMMA 2. *If $\mathbb{F}$ is finite and $V^{\mathbb{E}}$ is G-stable, and if the characteristic of $\mathbb{F}$ is the same as the index p of H in G, then there is a unique isomorphism class of $\mathbb{F}G$-modules U such that $U{\downarrow} \cong V$. The endomorphism field of U is $\mathbb{E}$, and $V{\uparrow}$ is uniserial of composition length p, with all composition factors isomorphic to U.*

Proof. The first sentence of this lemma was proved in Section 2, using that in a finite field of characteristic p every element has a unique pth root, and the first statement of the second sentence is also evident from the argument given there.

Towards proving the statement concerning $V{\uparrow}$, recall (for example from VII.4.15b in [2]) that, loosely speaking, $V{\uparrow} = U{\downarrow}{\uparrow} = U \otimes Y$ where Y is the regular $\mathbb{F}(G/H)$-module. Consider the 'outer tensor product' $U \sharp Y$, that is, $U \otimes Y$ viewed as a $(G \times (G/H))$-module: strictly speaking, the G-module $U \otimes Y$ is the restriction of $U \sharp Y$ along the 'diagonal embedding' $g \mapsto (g, gH)$ of G in $G \times (G/H)$. If $\langle c \rangle$ is a cyclic group of order p, then multiplication by $1 - c$ is an endomorphism of the regular $\mathbb{F}\langle c \rangle$-module, such that the images of the powers of this endomorphism form the unique composition series in that module. Thus Y has an endomorphism, θ say, such that the images of the powers of θ form a composition series with all factors 1-dimensional and trivial. Then $1 \otimes \theta$ is an endomorphism of $U \sharp Y$, and therefore also of $V{\uparrow}$. It follows that the images of $V{\uparrow}$ under the powers of $1 \otimes \theta$ form a composition series for $V{\uparrow}$ with all factors isomorphic to U. By Nakayama Reciprocity,

every $\mathbb{F}G$-module whose restriction to H is V must be a homomorphic image of $V{\uparrow}$; we have just seen that all composition factors of $V{\uparrow}$ are isomorphic to U; hence (up to isomorphism) U is the only $\mathbb{F}G$-module with $U{\downarrow} \cong V$. Nakayama Reciprocity also tells us that $\mathrm{Hom}_{\mathbb{F}G}(V{\uparrow}, U) \cong \mathrm{End}_{\mathbb{F}H} V$, and we already know that $\mathrm{End}_{\mathbb{F}H} V = \mathrm{End}_{\mathbb{F}G} U$, so it follows that there is only one submodule in $V{\uparrow}$ with quotient isomorphic to U: thus $V{\uparrow}$ has only one maximal submodule. Then each term of the given composition series of $V{\uparrow}$, being a homomorphic image of $V{\uparrow}$, has only one maximal submodule, and therefore this is the only composition series in $V{\uparrow}$. $\square$

Examples. The simplest examples are the cyclic groups G of order p. For an example with faithful U, take $\mathbb{F} = GF(3)$, $G = SL(2,3)$, and V the restriction of the natural $\mathbb{F}G$-module.

6. The absolutely stable case in characteristic different from p

Suppose now that the characteristic of $\mathbb{F}$ is not the index p of H in G. Recall that in this case we agreed to write $\mathbb{E}\big(\sqrt[p]{1}\big)$ for the field obtained from $\mathbb{E}$ by adjoining a primitive pth root of 1, and s for the degree of this field over $\mathbb{E}$; also, we noted that s is a divisor of $p-1$. For finite $\mathbb{F}$, parts (c) and (d) of the Theorem may be paraphrased as follows.

LEMMA 3. *Suppose that $\mathbb{F}$ is finite, $V^{\mathbb{E}}$ is G-stable, and the characteristic of $\mathbb{F}$ is not p.*

(c) *If $s > 1$, then there is a unique isomorphism type of $\mathbb{F}G$-modules U such that $U{\downarrow} \cong V$; the endomorphism field of U is $\mathbb{E}$, and*

$$V{\uparrow} \cong U \oplus W_1 \oplus \cdots \oplus W_{(p-1)/s}$$

where the W_j are pairwise nonisomorphic irreducible $\mathbb{F}G$-modules, each with endomorphism field $\mathbb{E}\big(\sqrt[p]{1}\big)$, each with $W_j{\downarrow} \cong V^{\oplus s}$.

(d′) *If $s = 1$ and if there exist $\mathbb{F}G$-modules U such that $U{\downarrow} \cong V$, then these U form p distinct isomorphism classes. If $U_1, \ldots, U_p$ are representatives*

of these isomorphism classes, then $V{\uparrow} \cong U_1 \oplus \cdots \oplus U_p$*, and each* U_j *has endomorphism field* $\mathbb{E}$*.*

(d″) *If* $s = 1$ *but there is no* $\mathbb{F}G$*-module* U *such that* $U{\downarrow} \cong V$*, then* $V{\uparrow}$ *is irreducible and its endomorphism field is the* (*unique*) *degree* p *extension of* $\mathbb{E}$*.*

Examples. In each of the examples to be given here, V can be any faithful irreducible $\mathbb{F}H$-module. For part (c), let $\mathbb{F}$ be $GF(2)$, G cyclic of order 45, and $p = 5$; for part (d′), let $\mathbb{F}$ be $GF(2)$ or $GF(4)$, G cyclic of order 63, and $p = 3$; for part (d″), let $\mathbb{F}$ be $GF(2)$ or $GF(4)$, G cyclic of order 9, and $p = 3$. In general, if $\mathbb{F} = GF(q)$, $\mathbb{F}\big(\sqrt[p]{1}\big) = GF(q^t)$, and k is any divisor of t other than 1, the cyclic group G of order $p(q^{t/k} - 1)$ gives an example of case (c) with $s = k$. [The example with G of order 45 given above was chosen a little more complicated to show also that $\mathbb{E}$ need not lie in $\mathbb{F}\big(\sqrt[p]{1}\big)$.] In parts (d′) and (d″), the two choices of $\mathbb{F}$ above illustrate that it is immaterial whether $\mathbb{F}$ itself contains a primitive pth root of 1.

In the light of the Section 2, it is easy to see that Lemma 3 is implied by the following two results.

Lemma 4. *If* $\mathbb{F}$ *is finite and* U *is an* $\mathbb{F}G$*-module such that* $U{\downarrow} = V$ *and* $\operatorname{End}_{\mathbb{F}G} U = \mathbb{E}$*, then* $V{\uparrow} \cong U \oplus W_1 \oplus \cdots \oplus W_{(p-1)/s}$ *where* U *and the* W_j *are pairwise nonisomorphic, and each* W_j *is irreducible with endomorphism field* $\mathbb{E}\big(\sqrt[p]{1}\big)$ *and* $W_j{\downarrow} \cong V^{\oplus s}$*.*

Lemma 5. *If* $\mathbb{F}$ *is finite and there is no* $\mathbb{F}G$*-module* U *such that* $U{\downarrow} \cong V$*, then* $V{\uparrow}$ *is irreducible and* $|\operatorname{End}_{\mathbb{F}G}(V{\uparrow}) : \mathbb{E}| = p$*.*

Proof of Lemma 4. As in the proof of Lemma 2, we use that $V{\uparrow} = U \otimes Y$ where Y is the regular $\mathbb{F}(G/H)$-module and $U \otimes Y$ is the restriction of the $\mathbb{F}(G \times (G/H))$-module $U \sharp Y$ along the diagonal embedding of G into $G \times (G/H)$. Write the image of that embedding as $\operatorname{diag} G$, noting that it is a normal subgroup (indeed, it is even a direct complement to $1 \times (G/H)$) in

$G \times (G/H)$, and that

$$(G \times 1) \cap (\operatorname{diag} G) = H \times 1 \qquad \text{and} \qquad (G \times 1)(\operatorname{diag} G) = G \times (G/H).$$

The restriction of $U \sharp Y$ to $G \times 1$ is just $U^{\oplus p}$. As $\operatorname{End}_{\mathbb{F}G} U = \operatorname{End}_{\mathbb{F}H}(U{\downarrow})$, we also have that $\operatorname{End}_{\mathbb{F}G}(U^{\oplus p}) = \operatorname{End}_{\mathbb{F}H}(U^{\oplus p}{\downarrow})$: that is, each $(H \times 1)$-endomorphism of $U \sharp Y$ is a $(G \times 1)$-endomorphism. In view of the last displayed equations, this implies that each $(\operatorname{diag} G)$-endomorphism of $U \sharp Y$ is a $(G \times (G/H))$-endomorphism: that is, $\operatorname{End}_{\mathbb{F}G}(U \otimes Y) = \operatorname{End}_{\mathbb{F}(G \times (G/H))}(U \sharp Y)$. We know that $\operatorname{End}_{\mathbb{F}G} U = \mathbb{E}$, while $\operatorname{End}_{\mathbb{F}(G/H)} Y = \mathbb{F}(G/H)$ because the endomorphism ring of the regular module for an abelian group is always the group algebra itself. Thus

$$\begin{aligned}
\operatorname{End}_{\mathbb{F}G}(V{\uparrow}) &= \operatorname{End}_{\mathbb{F}G}(U \otimes Y) \\
&= \operatorname{End}_{\mathbb{F}(G \times (G/H))}(U \sharp Y) \\
&= (\operatorname{End}_{\mathbb{F}G} U) \otimes_{\mathbb{F}} (\operatorname{End}_{\mathbb{F}(G/H)} Y) \quad \text{by VII.9.16b in [2]} \\
&= \mathbb{E} \otimes_{\mathbb{F}} \mathbb{F}(G/H) \\
&= \mathbb{E}(G/H) \\
&= \mathbb{E} \oplus \mathbb{E}\big(\sqrt[p]{1}\big) \oplus \cdots \oplus \mathbb{E}\big(\sqrt[p]{1}\big) \qquad \text{with } 1 + (p-1)/s \text{ summands}
\end{aligned}$$

where the last line comes from the known structure of group algebras of cyclic groups. By VII.9.4 of [2], $V{\uparrow}$ is completely reducible. The submodule structure of a completely reducible module can always be read off the endomorphism ring of the module. In the present case the conclusion may be put as follows: $V{\uparrow}$ is a direct sum of $1 + (p-1)/s$ pairwise nonisomorphic irreducibles, one with endomorphism field $\mathbb{E}$, each of the others with endomorphism field $\mathbb{E}\big(\sqrt[p]{1}\big)$.

We know that U is one of the direct summands of $V{\uparrow}$; denote the other irreducible direct summands by $W_1, \ldots, W_{(p-1)/s}$, so $\operatorname{End}_{\mathbb{F}G} W_j = \mathbb{E}\big(\sqrt[p]{1}\big)$. From $V{\uparrow}{\downarrow} = V^{\oplus p}$ we see that $W_j{\downarrow} \cong V^{\oplus t(j)}$ with integers $t(j)$ such that $\sum t(j) = p - 1$. What remains to show is that $t(1) = \cdots = t((p-1)/s) = s$.

The endomorphism field of W_j is a subring in the endomorphism ring of $W_j{\downarrow}$, and the latter is the $t(j) \times t(j)$ matrix ring over $\mathbb{E}$. A minimal right ideal of that matrix ring may therefore be viewed as a vector space over $\mathbb{E}\big(\sqrt[p]{1}\big)$, so $t(j)$ must be a multiple of s. Here all we need from this is that $t(j) \geq s$, for then $\sum t(j) = p - 1$ implies that each $t(j)$ is equal to s. $\square$

Proof of Lemma 5. Let $\mathbb{E}^p$ denote the (unique) degree p extension of $\mathbb{E}$. If $V^{\mathbb{E}}$ were the restriction of an $\mathbb{E}G$-module X, then $X_{\mathbb{F}}$ could serve as the U whose non-existence is assumed: thus there can be no such X. Accordingly, the second last paragraph of Section 2 can be applied with $\mathbb{E}$ and $V^{\mathbb{E}}$ in place of $\mathbb{F}$ and V: there exist $\mathbb{E}^pG$-modules $X_1, \ldots, X_p$ that are pairwise nonisomorphic but $\mathrm{Gal}\big(\mathbb{E}^p\big|\mathbb{E}\big)$-conjugate and such that each $X_i{\downarrow}$ is $V^{\mathbb{E}} \otimes_{\mathbb{E}} \mathbb{E}^p$. It follows by reciprocity that $(V^{\mathbb{E}} \otimes_{\mathbb{E}} \mathbb{E}^p){\uparrow}$ has a homomorphism onto $\bigoplus X_i$, and then by dimension comparison $(V^{\mathbb{E}} \otimes_{\mathbb{E}} \mathbb{E}^p){\uparrow} \cong \bigoplus X_i$. The point we need from this is that no proper nonzero submodule of $(V^{\mathbb{E}} \otimes_{\mathbb{E}} \mathbb{E}^p){\uparrow}$ can have $\mathrm{Gal}\big(\mathbb{E}^p\big|\mathbb{E}\big)$-invariant isomorphism type.

Of course $(V^{\mathbb{E}} \otimes_{\mathbb{E}} \mathbb{E}^p){\uparrow} = (V^{\mathbb{E}}{\uparrow}) \otimes_{\mathbb{E}} \mathbb{E}^p$. If X is any submodule of $V^{\mathbb{E}}{\uparrow}$, then $X \otimes_{\mathbb{E}} \mathbb{E}^p$ is a submodule of $(V^{\mathbb{E}}{\uparrow}) \otimes_{\mathbb{E}} \mathbb{E}^p$ and the isomorphism type of $X \otimes_{\mathbb{E}} \mathbb{E}^p$ is certainly $\mathrm{Gal}\big(\mathbb{E}^p\,\big|\,\mathbb{E}\big)$-invariant. Thus X cannot be proper and nonzero: $V^{\mathbb{E}}{\uparrow}$ must be irreducible. Note, however, that $V^{\mathbb{E}}{\uparrow}$ is not absolutely irreducible.

When an irreducible $\mathbb{E}G$-module is viewed as an $\mathbb{F}G$-module, it is a direct sum of isomorphic irreducible summands (see VII.1.16d in [2]). As $(V^{\mathbb{E}}{\uparrow})_{\mathbb{F}} = V{\uparrow}$, therefore $V{\uparrow} = W^{\oplus t}$ for some irreducible W and some integer t. Because $(W{\downarrow})^{\oplus t} = V{\uparrow}{\downarrow} = V^{\oplus p}$ and p is prime, we can only have $t = 1$ or $t = p$. The second of these alternatives is excluded by the hypothesis that V is not the restriction of any $\mathbb{F}G$-module, and $t = 1$ means that $V{\uparrow}$ is irreducible, as required.

The endomorphism field of $V^{\mathbb{E}}{\uparrow}$ is a subring in the endomorphism ring of $V^{\mathbb{E}}{\uparrow}{\downarrow}$ which is the $p \times p$ matrix ring over $\mathbb{E}$, so (by an argument used in the last paragraph of the proof of Lemma 4) $|\operatorname{End}_{\mathbb{E}G}(V^{\mathbb{E}}{\uparrow}) : \mathbb{E}|$ is a divisor of p.

This degree cannot be 1, because we have seen that $V^{\mathbb{E}}\!\uparrow$ is not absolutely irreducible. Thus $\mathrm{End}_{\mathbb{E}G}(V^{\mathbb{E}}\!\uparrow) = \mathbb{E}^p$, and therefore also $\mathrm{End}_{\mathbb{F}G}(V\!\uparrow) = \mathbb{E}^p$. □

Lemmas 1, 2, 3 together prove the Theorem for finite $\mathbb{F}$.

Remark. If $\mathbb{F}$ is algebraically closed, cases (c) and (d″) do not arise in Lemma 3, and another theorem of Clifford (51.7 in Curtis and Reiner [1]) is also available: if $V = U\!\downarrow$ then the irreducible direct summands of $V\!\uparrow$ are precisely the modules obtained by tensoring U with the irreducible $\mathbb{F}(G/H)$-modules (viewed as $\mathbb{F}G$-modules with kernels containing H). It is not hard to see that in case (d′) this is true without any assumption on $\mathbb{F}$, and that it is true even in case (c) provided that $\mathbb{E} \cap \mathbb{F}\big(\sqrt[p]{1}\big) = \mathbb{F}$. The first of the examples mentioned after the statement of Lemma 3 shows that this proviso is necessary. (The way to avoid the proviso is to consider instead the $(U^{\mathbb{E}} \otimes_{\mathbb{E}} Y)_{\mathbb{F}}$ with Y ranging through the irreducible $\mathbb{E}(G/H)$-modules.)

7. Infinite fields

Restricting attention to finite fields in the preceding sections was convenient but not really necessary. In prime characteristic, the endomorphism ring of an irreducible module is always commutative; indeed, it is always a cyclic extension of the ground field obtained by adjoining a root of 1. This fact is tied up with a (presumably well-known) general dispensation to the effect that the theory of completely reducible representations of a finite group over a field of prime characteristic is always 'the same' as over a suitable finite subfield. In the absence of a convenient reference, we state and prove this as follows.

Given a field $\mathbb{F}$ and a finite group G, write $\mathrm{Irr}\,\mathbb{F}G$ for the set of all isomorphism types of irreducible (right) $\mathbb{F}G$-modules. Let n be a positive integer, $\mathbb{F}$ a field of prime characteristic, $\mathbb{F}^{(n)}$ the (finite) subfield of the algebraic closure of $\mathbb{F}$ generated by the roots of $x^n - 1$, and $\mathbb{F}_{(n)} = \mathbb{F}^{(n)} \cap \mathbb{F}$.

Lemma 6. *If G is a finite group of exponent dividing n, then $U \mapsto U \otimes_{\mathbb{F}_{(n)}} \mathbb{F}$*

is a bijection from $\operatorname{Irr}\mathbb{F}_{(n)}G$ *to* $\operatorname{Irr}\mathbb{F}G$.

We know (see VII.1.12 in [2]) that $\operatorname{End}_{\mathbb{F}G}(U\otimes_{\mathbb{F}_{(n)}}\mathbb{F}) = (\operatorname{End}_{\mathbb{F}_{(n)}G}U)\otimes_{\mathbb{F}_{(n)}}\mathbb{F}$, thus Lemma 6 implies that the endomorphism ring of each irreducible $\mathbb{F}G$-module is a commutative field obtained by adjoining to $\mathbb{F}$ a root of 1. It is clear that the bijections described in Lemma 6 commute with induction and restriction. It follows that *the earlier lemmas of this paper hold not only for finite* $\mathbb{F}$ *but for all* $\mathbb{F}$ *of nonzero characteristic.* There is one slight change of wording necessary: in part (d″) of Lemma 3, the endomorphism field of $V\uparrow$ has to be identified as the unique degree p extension of $\mathbb{E}$ *obtainable by adjoining roots of* 1. (Note that no field can ever have more than one degree p extension of this kind. When $\mathbb{E}$ has no such extension, we cannot be in case (d″) of Lemma 3. On the other hand, regardless of whether $\mathbb{E}$ has such an extension, if $\mathbb{E}$ is not algebraic over its prime subfield then it may also have degree p extensions whose finite subfields are all contained in $\mathbb{E}$.) In the case of Lemma 2, in order to see that when V is an irreducible $\mathbb{F}_{(n)}H$-module such that $V\uparrow$ is uniserial then $(V\otimes_{\mathbb{F}_{(n)}}\mathbb{F})\uparrow$ is also uniserial, we cannot rely simply on Lemma 6 and Lemma 2. Instead, we have to argue that, given the rest of Lemma 2, the proof of uniseriality never used the finiteness assumption. As the Lemmas 1, 2, 3 together proved the finite fields case of the Theorem, we have now proved the Theorem in its full generality.

For the *proof of Lemma* 6, it suffices to show that $(\operatorname{End}_{\mathbb{F}_{(n)}G}U)\otimes_{\mathbb{F}_{(n)}}\mathbb{F}$ is a field. Indeed, $U\otimes_{\mathbb{F}_{(n)}}\mathbb{F}$ is completely reducible (by VII.1.8 of [2]), so if its endomorphism ring is a field then it must be irreducible. Given this, the map defined in the lemma is one-to-one by VII.1.22 of [2], and surjective because, by VII.1.5 of [2], the largest semisimple quotient of the group algebra $\mathbb{F}G$ may be obtained from that of $\mathbb{F}_{(n)}G$ by tensoring over $\mathbb{F}_{(n)}$ with $\mathbb{F}$.

By VII.2.6 in Huppert and Blackburn [2], $\mathbb{F}^{(n)}$ is a splitting field for G. Let $U\in\operatorname{Irr}\mathbb{F}_{(n)}G$. The irreducible direct summands of $U\otimes_{\mathbb{F}_{(n)}}\mathbb{F}^{(n)}$ are all absolutely irreducible and therefore their endomorphism fields are all equal to $\mathbb{F}^{(n)}$. The direct sum of these endomorphism fields is $\operatorname{End}_{\mathbb{F}^{(n)}G}(U\otimes_{\mathbb{F}_{(n)}}\mathbb{F}^{(n)})$,

and in turn this is $(\mathrm{End}_{\mathbb{F}_{(n)}G} U) \otimes_{\mathbb{F}_{(n)}} \mathbb{F}^{(n)}$. We see from VII.1.4b of [2] that the latter algebra can only be a direct sum of copies of $\mathbb{F}^{(n)}$ if $\mathrm{End}_{\mathbb{F}_{(n)}G} U$ is isomorphic to a subfield of $\mathbb{F}^{(n)}$. We conclude that the unique copy of $\mathrm{End}_{\mathbb{F}_{(n)}G} U$ in the algebraic closure of $\mathbb{F}$ intersects $\mathbb{F}$ precisely in $\mathbb{F}_{(n)}$. In turn, this implies that $(\mathrm{End}_{\mathbb{F}_{(n)}G} U) \otimes_{\mathbb{F}_{(n)}} \mathbb{F}$ is a field, as required. $\square$

We conclude by noting that over fields of characteristic 0 that are not algebraically closed, the general picture is quite different. The proofs we used break down, and some of the conclusions we reached make no sense (because the endomorphism ring of an irreducible need not be a field). Worse still, even conclusions that would make sense can fail to hold. For example, the Theorem implies that in prime characteristic $V{\uparrow}$ is either multiplicity-free (that is, a direct sum of pairwise nonisomorphic irreducibles) or homogeneous (in the sense of all compositions factors having the same isomorphism type). By contrast, if $\mathbb{F}$ is a subfield of the field of real numbers, the representation of $SL(2,3)$ induced from the unique faithful irreducible (4-dimensional) representation of Q_8 over $\mathbb{F}$ involves two irreducibles of $SL(2,3)$, one with multiplicity 1 and the other with multiplicity 2.

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References

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[2] B. Huppert, N. Blackburn, *Finite Groups II*, Springer-Verlag, Berlin, 1982.